Part IA Paper 4: Mathematical Methods

Examples Paper 8

Elementary exercises are marked †, problems of Tripos standard ∗.
Hints and answers can be found at the back of the paper.

Fourier series

1. † A train of digital pulses is periodic with period 2π and has the form

$$f(t) = \begin{cases} 1 & 0 < t \leq T \\ 0 & T < t \leq 2\pi \end{cases}$$

   Express $f(t)$ as a Fourier series, and evaluate the coefficients.

   During transmission by a long cable, high-frequency components of the signal are attenuated. Explain briefly how the Fourier series allows this low-pass filtering effect to be studied.

   Now consider a specific example of a cable that transmits perfectly all frequency components below 1 kHz but attenuates completely all frequency components above 1 kHz. The digital pulse train has period 2π ms and $T = \pi$ ms. Use the Matlab/Octave code described in the lecture notes[1] to plot the filtered signal.

2. A triangular wave of period 2π has the form

$$f(\theta) = \begin{cases} \theta & 0 \leq \theta < \pi \\ 2\pi - \theta & \pi \leq \theta < 2\pi \end{cases}$$

   Explain why the Fourier series of $f(\theta)$ contains cosines but no sines, and evaluate the Fourier coefficients. Verify that the series for the square wave, derived in lectures, is the derivative of this series for the triangular wave.

3. A periodic function of period 2π has the form

$$f(\theta) = 2[\delta(\theta) - \delta(\theta - \pi)] \text{ for } -\pi/2 \leq \theta \leq 3\pi/2$$

   Express $f(\theta)$ as an appropriate Fourier series and evaluate the Fourier coefficients. Verify that the series for the square wave, derived in lectures, is the integral of this series.

   (Note that this is the *easiest* way to obtain the Fourier series of the square wave — differentiate it to give delta functions, so that the integrals for the Fourier coefficients are trivial, then integrate the answer.)

4. A full-wave rectified sine wave of angular frequency ω has the form

$$f(t) = |\sin(\omega t)|$$

   Sketch the function. Express it as a Fourier series, and calculate the coefficients.

[1] Available from `http://www-sigproc.eng.cam.ac.uk/~jl/index_IA_Maths.html`

5. The function
$$y(x) = \begin{cases} x(\pi + x) & -\pi \le x \le 0 \\ x(\pi - x) & 0 \le x \le \pi \end{cases}$$
is represented by a Fourier series of period 2π. Is dy/dx continuous? Is d^2y/dx^2 continuous? How do you expect the coefficients of the Fourier series to vary with n when n is large? Verify your answer to this last question by evaluating the coefficients.

6. $*$ A guitar string of length L is stretched between the points $x = 0$ and $x = L$. The *modes of vibration* of the string have the form
$$w_n(x) = \sin \frac{n\pi x}{L}$$
When a string is plucked, it is pulled into a displaced shape $y(x)$ and then released. Each vibration mode then produces sounds at its natural frequency. Thus the frequency content of the guitar sound is determined by the relative amplitudes of each mode excited by the pluck, in other words by the values b_n in the expression
$$y(x) = \sum_{n=1}^{\infty} b_n w_n(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$
Evaluate these Fourier coefficients for an idealised pluck at the point $x = a$ in which
$$y(x) = \begin{cases} kx & 0 \le x \le a \\ \dfrac{ka(L-x)}{L-a} & a \le x \le L \end{cases}$$
Sketch graphs of $|b_n|$ against n for the cases (i) $a = L/2$; (ii) $a = L/5$; (iii) $a = L/50$, and comment on the likely effect on the sound of the guitar of varying the plucking position.

In a more realistic model for a pluck, the sharp corner at $x = a$ would be rounded off over a short distance. Without any calculations, comment on the influence this will have on the behaviour of the Fourier coefficients at high mode numbers n.

7. $*$ The three Fourier series
(i) $\sum_{n=1}^{\infty} a_n \sin 2nx$; (ii) $\sum_{n=1}^{\infty} b_n \sin(2n-1)x$; (iii) $\sum_{n=0}^{\infty} c_n \cos 2nx$
all represent the function $f(x) = x^2$ in the range $0 \le x \le \pi/2$. Sketch the functions defined by these three series in the range $-\pi \le x \le \pi$. Explain why one of the series is expected to converge much more slowly than the others. (There is no need to evaluate the coefficients in any of the series.)

8. A function $y(t)$ having period T is defined as
$$y(t) = \begin{cases} e^{-\alpha t} & 0 < t \le T/2 \\ 0 & T/2 < t \le T \end{cases}$$
Obtain the coefficients C_n in the complex Fourier series for $y(t)$, where
$$y(t) = \sum_{n=-\infty}^{\infty} C_n e^{2n\pi i t/T}$$

Hence obtain the coefficients a_n and b_n in the real Fourier series for $y(t)$, where

$$y(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi t}{T} + \sum_{n=1}^{\infty} b_n \sin\frac{2n\pi t}{T}$$

(Notice that this is easier than calculating a_n and b_n directly.)

The material on this examples paper has been included in the mathematics course since 1993, and further questions can be found in Part IA Paper 4 in each year since then.

Timing of lectures
Questions 1–2 can be done after the lectures have covered Section 5 of the handouts (roughly mid February 2014)
Question 3 and the first part of question 4 can be done after the lectures have covered Section 6 of the handouts (roughly towards the end of February 2014)
The rest of question 4 and questions 5–7 can be done after the lectures have covered Section 7 of the handouts (roughly the beginning of March 2014
Question 8 can be done after the lectures have covered Section 8 of the handouts (first week in March 2014)

Hints

1. When $T = \pi$ ms, the digital pulse train is just a square wave and we can use the supplied code to study its Fourier series. Its fundamental frequency is $1000/2\pi = 159$ Hz and its sixth harmonic is $159 \times 6 = 955$ Hz. So the cable will pass the first six terms of the Fourier series and attenuate the rest. To view the filtered signal, we need only change line 5 of the program to `nharm = 6`.

   Try modifying the program to plot the Fourier series for other functions in this examples paper. For functions of period 2π, you need only change the expressions for d, a_n and b_n. For other functions, you could just force the period to 2π, i.e. set $\omega = 1$ in question 4, $L = \pi$ in question 6 and $T = 2\pi$ in question 8.

Answers

1. $f(t) = \dfrac{T}{2\pi} + \displaystyle\sum_{n=1}^{\infty} \frac{\sin nT}{n\pi} \cos nt + \sum_{n=1}^{\infty} \frac{(1-\cos nT)}{n\pi} \sin nt$

2. $f(\theta) = \dfrac{\pi}{2} - \dfrac{4}{\pi}\left[\cos\theta + \dfrac{1}{3^2}\cos 3\theta + \dfrac{1}{5^2}\cos 5\theta + \ldots\right]$

3. $f(\theta) = \dfrac{4}{\pi}\left[\cos\theta + \cos 3\theta + \cos 5\theta + \ldots\right]$

4. $f(t) = \dfrac{2}{\pi} - \dfrac{4}{\pi}\left[\dfrac{1}{3}\cos 2\omega t + \dfrac{1}{15}\cos 4\omega t + \dfrac{1}{35}\cos 6\omega t + \ldots\right]$

5. dy/dx continuous, d^2y/dx^2 discontinuous, coefficients of order $1/n^3$

   $y(x) = \dfrac{8}{\pi}\left[\sin x + \dfrac{1}{3^3}\sin 3x + \dfrac{1}{5^3}\sin 5x + \ldots\right]$

6. $b_n = \dfrac{2kL^2}{n^2\pi^2(L-a)} \sin\dfrac{n\pi a}{L}$

8. $C_n = \dfrac{1 - e^{-\alpha T/2 - in\pi}}{\alpha T + 2in\pi}$

   $a_0 = \dfrac{2(1-e^{-\alpha T/2})}{\alpha T}$, $a_n = \dfrac{2\alpha T\left(1-(-1)^n e^{-\alpha T/2}\right)}{\alpha^2 T^2 + 4n^2\pi^2}$, $b_n = \dfrac{4n\pi\left(1-(-1)^n e^{-\alpha T/2}\right)}{\alpha^2 T^2 + 4n^2\pi^2}$

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