3F1 Information Theory, Lecture 3

Jossy Sayir

University of Cambridge
Department of Engineering

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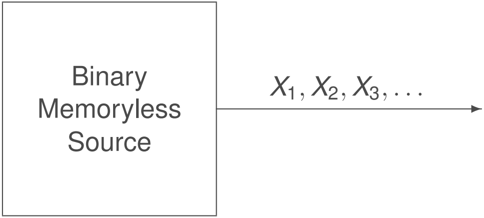
Encoding the output of a source



- $X_1, X_2, \ldots$ independent and identically distributed
- $P_{X_i}(1) = 1 - P_{X_i}(0) = 0.01$
- What's the optimal binary variable length code for X_i?
- $E[W] = 1$, but $H(X) = 0.081$ bits.
- Redundancy $\rho_1 = E[W] - H(X) = 0.9190$
- Can we do better?

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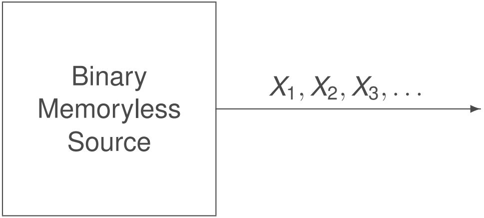

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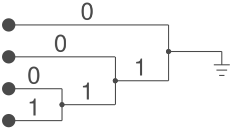
Parsing a source into blocks of 2

x_1x_2	$P_{X_1X_2}(x_1,x_2)$
00	0.9801
01	0.0099
10	0.0099
11	0.0001



Codeword
0
10
110
111

- $E[W] = 1.0299$, $H(X_1X_2) = 0.1616$ bits, where

$$H(X_1X_2) = -\sum_{x_1x_2} P_{X_1X_2}(x_1,x_2)\log P_{X_1X_2}(x_1,x_2)$$

- Redundancy per symbol:

$$\rho_2 = \frac{E[W] - H(X_1X_2)}{2} = 0.4342$$

- Can we do better?

Encoding a block of length N

- If encoding a block of length N using Huffman or Shannon-Fano coding,

$$H(X_1 \dots X_N) \leq E[W] < H(X_1 \dots X_N) + 1$$

where

$$H(X_1 \dots X_2) = - \sum_{x_1 \dots x_2} P(x_1, \dots, x_2) \log P(x_1, \dots, x_2)$$

- Thus,

$$\rho_N = \frac{E[W] - H(X_1 \dots X_N)}{N} \leq \frac{H(X_1 \dots X_N) + 1 - H(X_1 \dots X_N)}{N} = \frac{1}{N}$$

and

$$\lim_{N \to \infty} \rho_N = 0,$$

i.e., the redundancy tends to zero

The problem with block compression

- The alphabet size grows exponentially in the block length N
- Huffman's and Fano's algorithms become infeasible for large N, as does storing the codebook
- In Shannon's version of Shannon-Fano coding, the probability and cumulative probability can be computed recursively:

$$P(x_1,\ldots,x_N) = P(x_1,\ldots,x_{N-1})P(x_N)$$
$$F(x_1,\ldots,x_N) = F(x_1,\ldots,x_{N-1}) + F(x_N)P(x_1,\ldots,x_{N-1})$$

- Compute the cumulative probability for a specific block without computing all others!
- If only there wasn't the need for the alphabet to be ordered...

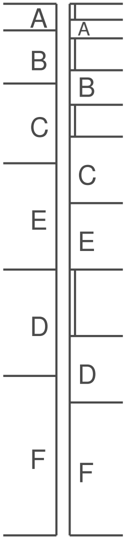
Reminder: Shannon's version of S-F Coding

x	$P_X(x)$	$P(X<x)$	$P(X < x)\|_b$	$\lceil -\log P_X(x) \rceil$	Codeword
F	.30	0.0	0.00000000	2	00
D	.20	0.3	0.01001101	3	010
E	.20	0.5	0.10000000	3	100
C	.15	0.7	0.10110011	3	101
B	.10	0.85	0.11011010	4	1101
A	.05	0.95	0.11110011	5	11110

- Order the symbols in order of non-increasing probability
- Compute the cumulative probabilities
- Express the cumulative probabilities in D-ary
- The codeword is the fractional part of the cumulative probabilities truncated to length $\lceil -\log P_X(x) \rceil$

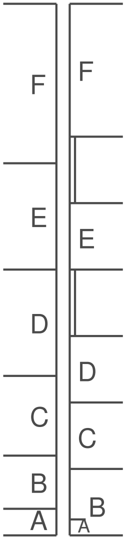
Source intervals / code intervals

x	$P(X<x)$	$P(X<x)\vert_b$
F	0.0	0.00
D	0.3	0.010
E	0.5	0.100
C	0.7	0.101
B	0.85	0.1101
A	0.95	0.11110



What if we don't order the source probabilities?

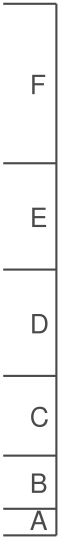
x	$P(X<x)$	$P(X<x)\|_b$
A	0.0	0.00000
B	0.05	0.000
C	0.15	0.001
D	0.3	0.010
E	0.5	0.100
F	0.7	0.11



New approach

x	$[a, b]$
A	$[0.0, 0.05]$
B	$[0.05, 0.15]$
C	$[0.15, 0.3]$
D	$[0.3, 0.5]$
E	$[0.5, 0.7]$
F	$[0.7, 1.0]$

$$a = P(X < x)$$
$$b = P(X < x) + P(X = x)$$



- Draw source intervals in no particular order
- Pick the largest interval $[k2^{-w_i}, (k+1)2^{-w_i}]$ that fits in each source interval
- How large is w_i?
- $w_i \leq \lceil -\log p_i \rceil + 1$

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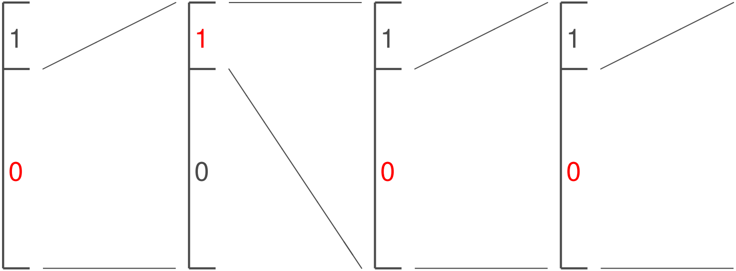
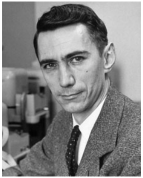
Arithmetic Coding

- Recursive computation of the source interval:

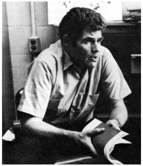


- At the end of the source block, generate the shortest possible codeword whose interval fits in the computed source interval
- $E[W] < H(X_1 \ldots X_N) + 2$

Arithmetic Coding = Recursive Shannon-Fano Coding


Claude E. Shannon


Robert L. Fano


Peter Elias


Richard C. Pasco


Jorma Rissanen

- $H(X_1 \dots X_N) \leq E[W] < H(X_1 \dots X_N) + 2$
- $\rho_N = 2/N$
- No need to wait until all the source block has been received to start generating code symbols
- Arithmetic Coding is an infinite state machine whose states need ever growing precision
- Finite precision implementation requires a few tricks (and loses some performance)

English Text

Letter	Frequency	Letter	Frequency	Letter	Frequency
a	0.08167	j	0.00153	s	0.06327
b	0.01492	k	0.00772	t	0.09056
c	0.02782	l	0.04025	u	0.02758
d	0.04253	m	0.02406	v	0.00978
e	0.12702	n	0.06749	w	0.02360
f	0.02228	o	0.07507	x	0.00150
g	0.02015	p	0.01929	y	0.01974
h	0.06094	q	0.00095	z	0.00074
i	0.06966	r	0.05987		

- $H(X) = 4.17576$ bits
- $P(''and'') = 0.08167 \times 0.06749 \times 0.04253 = 0.0002344216$
- $P(''eee'') = 0.12702^3 = 0.00204935$
- $P(''eee'') >> P(''and'')$. Is this right? No!
- Can we do better than $H(X)$ for sources with memory?

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Discrete Stationary Source (DSS) properties

- $H_N(X) \stackrel{\text{def}}{=} \frac{1}{N} H(X_1 \dots X_N)$

Entropy Rate

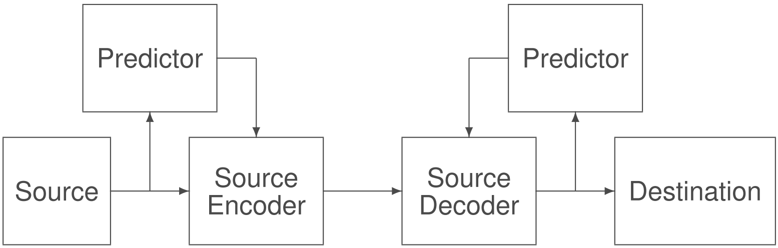
- $H(X_N | X_1 \dots X_{N-1}) \leq H_N(X)$
- $H(X_N | X_1 \dots X_{N-1})$ and $H_N(X)$ are non-increasing functions of N
- $\lim_{N\to\infty} H(X_N | X_1 \dots X_{N-1}) = \lim_{N\to\infty} H_N(X) \stackrel{\text{def}}{=} H_\infty(X)$
- $H_\infty(X)$ is the entropy rate of a DSS

Shannon's converse source coding theorem for a DSS

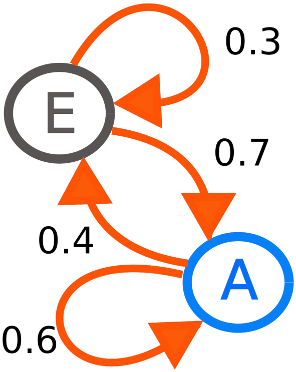
$$\frac{E[W]}{N} \geq \frac{H_\infty(X)}{\log D}$$

Coding for discrete stationary sources

- Arithmetic coding can use conditional probabilities
- The intervals will be different at every step depending on the source context
- "Prediction by Partial Matching" (PPM) and "Context Tree Weighing" (CTW) are techniques to build the context tree based source model on the fly, achieving compression rates of approx 2.2 binary symbols per ASCII character
- What is $H_\infty(X)$ for English text? (assuming language is a stationary source, which is a disputed proposition)



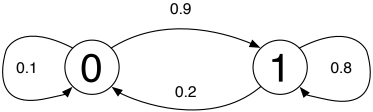
Markov Chain



Andrey Andreyevivich Markov

- Stationary state random process $S_1, S_2, \ldots$
- $P(s_N | s_1 \ldots s_{N-1}) = P(s_N | s_{N-1})$
- Markov information source: states S_i are mapped into source symbols X_i
- Unifilar information source: from any state, all neighbouring states map to distinct symbols

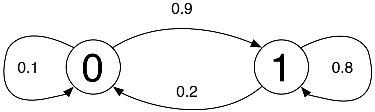
Unifilar Markov Source



- $P_{X_2|X_1}(1|0) = 1 - P_{X_2|X_1}(0|0) = 0.9$
- $P_{X_2|X_1}(1|1) = 1 - P_{X_2|X_1}(0|1) = 0.8$
- Can we compute $P_{X_1}(1) = 1 - P_{X_1}(0)$?
- Stationarity implies $P_{X_1}(1) = P_{X_2}(1)$ and thus

$$\begin{aligned} P_{X_1}(1) = P_{X_2}(1) &= P_{X_1X_2}(01) + P_{X_1X_2}(11) \\ &= P_{X_2|X_1}(1|0)P_{X_1}(0) + P_{X_2|X_1}(1|1)P_{X_1}(1) \end{aligned}$$

Unifilar Markov Source



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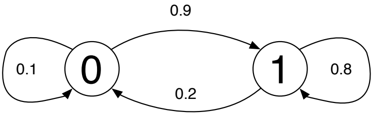


- Define the matrix

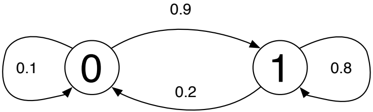
$$T = \begin{bmatrix} P_{X_2|X_1}(0|0) & P_{X_2|X_1}(0|1) \\ P_{X_2|X_1}(1|0) & P_{X_2|X_1}(1|1) \end{bmatrix}$$

and the vector $P = [P_{X_1}(0), P_{X_1}(1)]^T$, then we are looking for the solution P to the equation

$$P = TP,$$

i.e., the eigenvector of T for the eigenvalue 1. Note that since T is a stochastic matrix (its columns sum to 1), it will always have 1 as an eigenvalue.

Unifilar Markov Source



- $P = \begin{bmatrix} 0.1 & 0.2 \\ 0.9 & 0.8 \end{bmatrix}$ P implies

$$\begin{bmatrix} -0.9 & 0.2 \\ 0.9 & -0.2 \end{bmatrix} P = 0$$

which, together with the constraint $[11]P = 1$ (probabilities sum to 1) yields

$$\begin{bmatrix} -0.9 & 0.2 \\ 1 & 1 \end{bmatrix} P = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

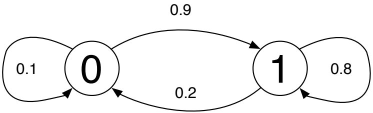
and finally

$$P = \begin{bmatrix} P_{X_1}(0) \\ P_{X_1}(1) \end{bmatrix} = \begin{bmatrix} 0.1818 \\ 0.8182 \end{bmatrix}$$

- Entropy rate of the source:

$$\begin{aligned} H_\infty(X) &= \lim_{N\to\infty} H(X_N|X_1 \ldots X_{N-1}) = H(X_N|X_{N-1}) = H(X_2|X_1) \\ &= H(X_2|X_1 = 0)P_{X_1}(0) + H(X_2|X_1 = 1)P_{X_1}(1) \\ &= 0.1818h(0.1) + 0.8182h(0.2) = 0.6759 \text{ bits} \end{aligned}$$

Encoding a unifilar Markov Source



- Encode source output sequence: 0,1,1,1,1,1,1,1

1.0

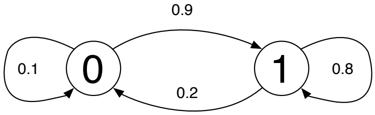
"1"

0.1818

"0"

0.0

Encoding a unifilar Markov Source



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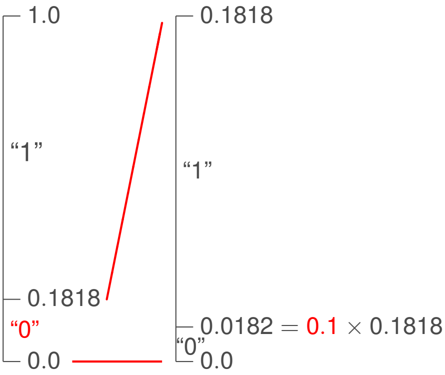

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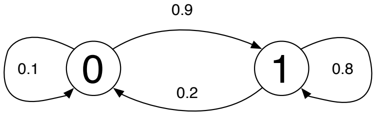

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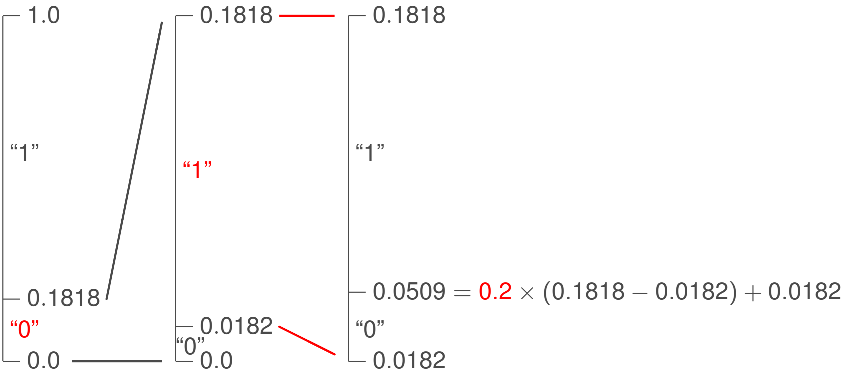

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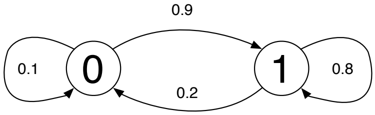

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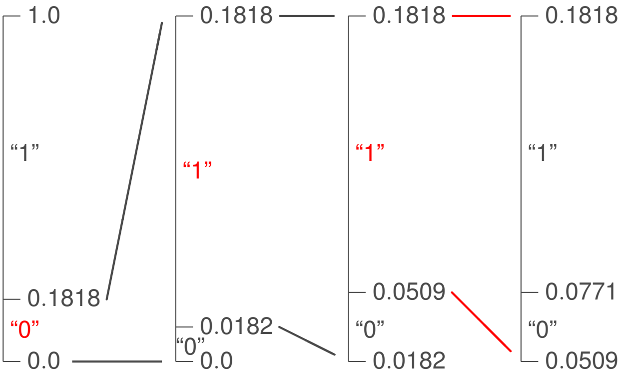

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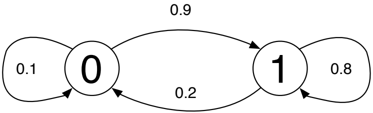

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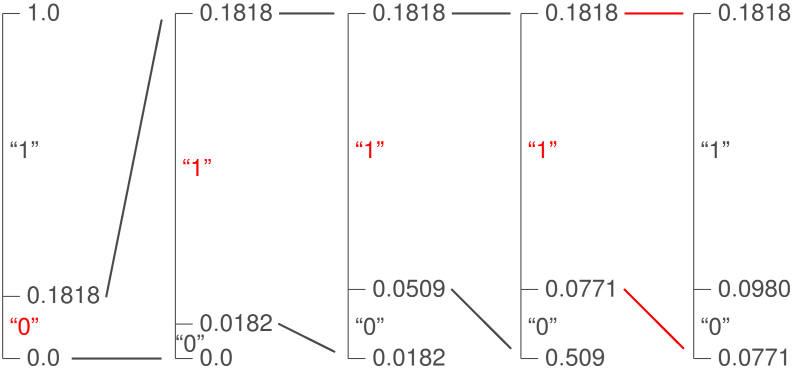

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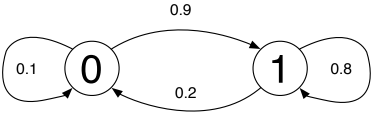

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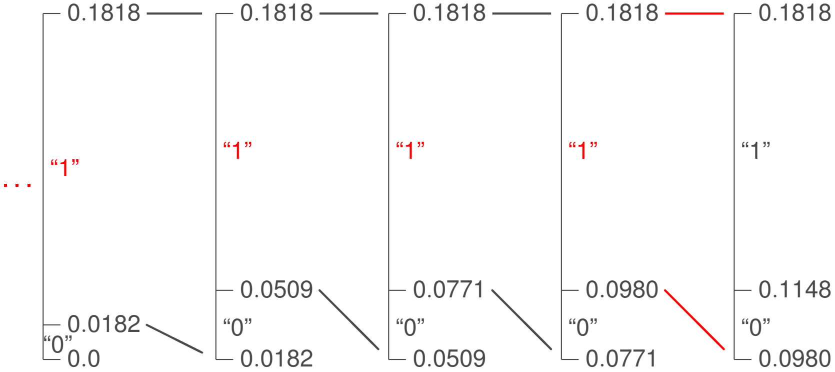

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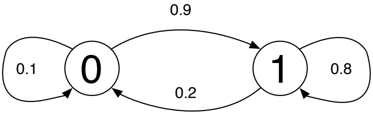

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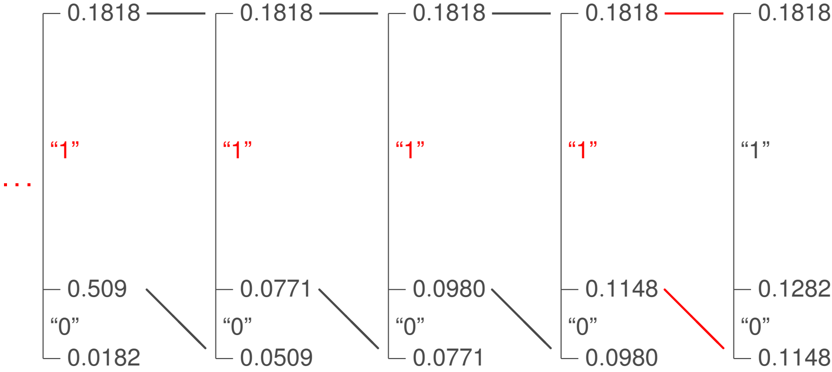

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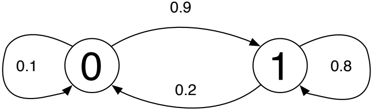

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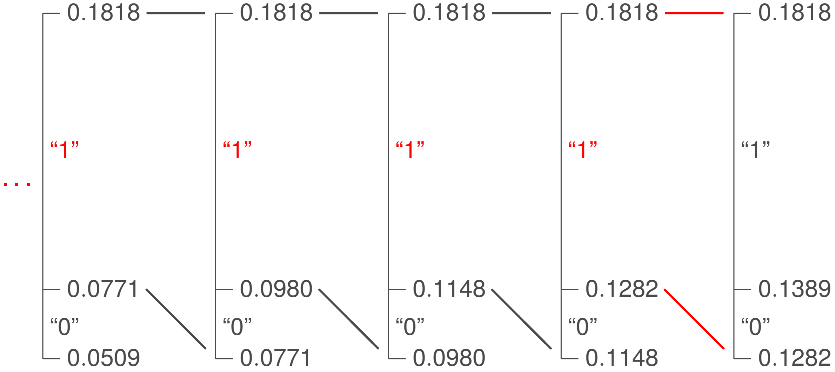

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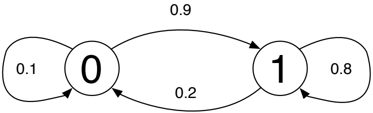

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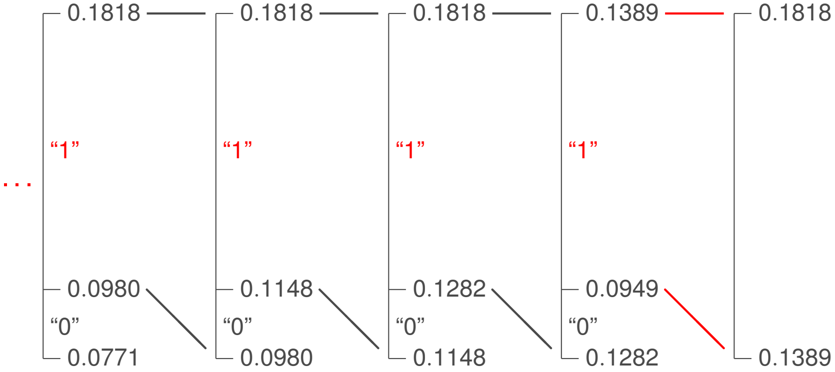

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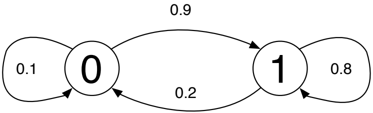

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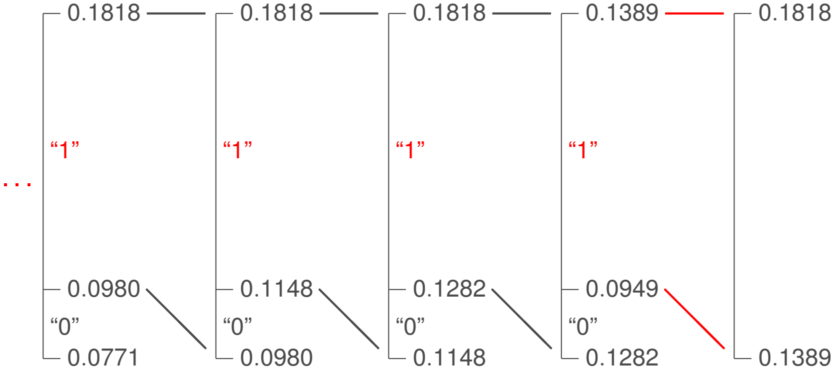


Determining the codeword

- Source interval $[0.1389, 0.1818]$ in binary:

$$[0.00100011, 0.00101110]_b$$

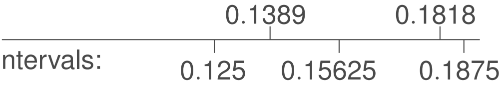
- The probability of the source sequence is

$$P_{X_1\ldots X_8}(0,1,1,1,1,1,1,1) = 0.1818 - 0.1389 = 0.042896$$

- $-\log_2 P_{X_1\ldots X_8}(0,1,1,1,1,1,1,1) = 4.543$, therefore we can either truncate after 5 or 6 digits, depending if the resulting code sequence is contained in the source interval
- No 5 digit code sequence corresponds to a code interval contained in our source interval:

Source interval:

Length 5 codeword intervals:

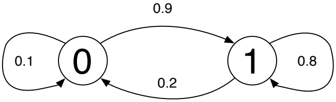


- The 6 digit code sequence 001010 corresponds to the code interval

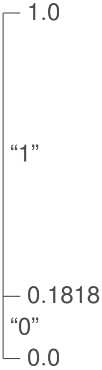
$$[0.001010, 0.001011]_b = [0.15625, 0.171875]$$

which is fully contained in the source interval and therefore satisfies the prefix condition

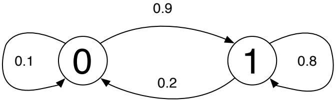
Decoding a unifilar Markov Source



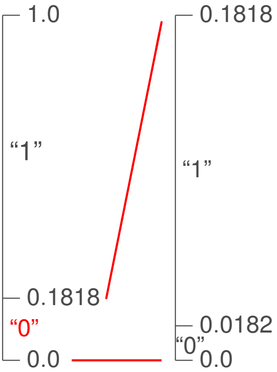
- Decode code sequence: 0,0,1,0,1,0 corresponding to interval $[0.15625, 0.171875]$
- Decoding rule: always pick sub-interval that contains the codeword interval



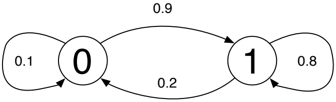
Decoding a unifilar Markov Source



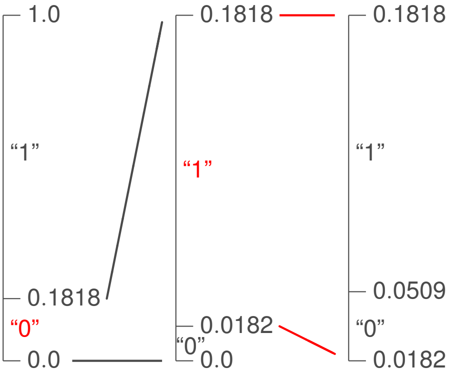
- Decode code sequence: 0,0,1,0,1,0 corresponding to interval $[0.15625, 0.171875]$
- Decoding rule: always pick sub-interval that contains the codeword interval, result: 0



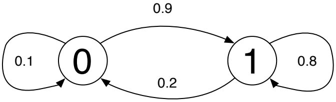
Decoding a unifilar Markov Source



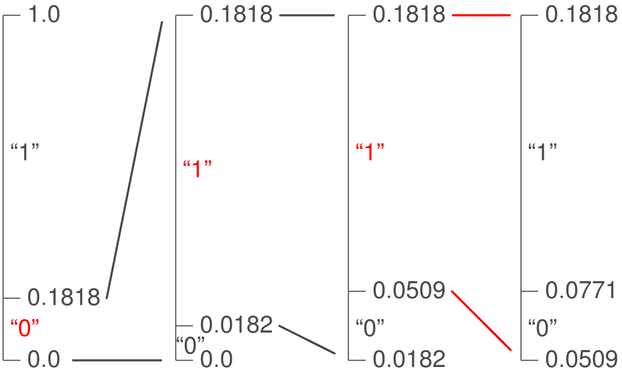
- Decode code sequence: 0,0,1,0,1,0 corresponding to interval $[0.15625, 0.171875]$
- Decoding rule: always pick sub-interval that contains the codeword interval, result: 0,1



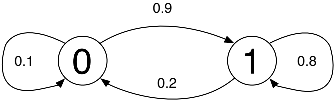
Decoding a unifilar Markov Source



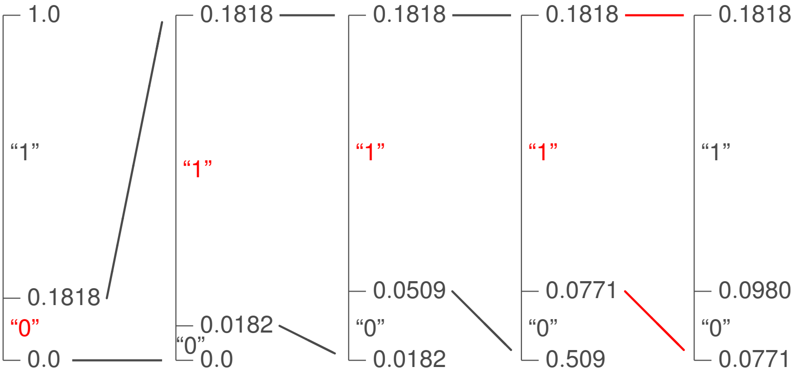
- Decode code sequence: 0,0,1,0,1,0 corresponding to interval $[0.15625, 0.171875]$
- Decoding rule: always pick sub-interval that contains the codeword interval, result: 0,1,1



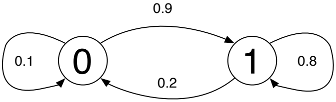
Decoding a unifilar Markov Source



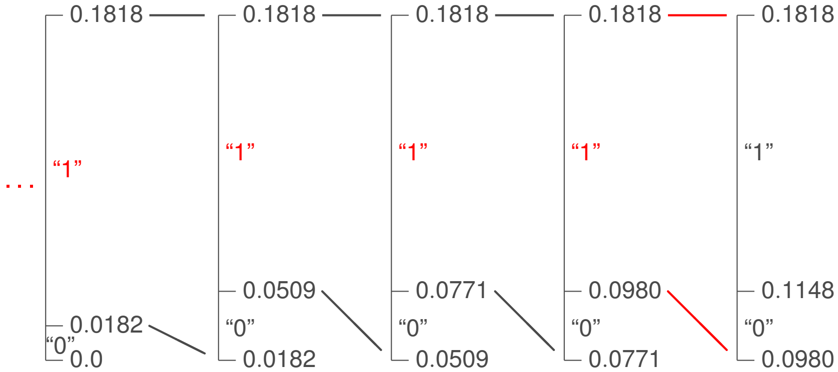
- Decode code sequence: 0,0,1,0,1,0 corresponding to interval $[0.15625, 0.171875]$
- Decoding rule: always pick sub-interval that contains the codeword interval, result: 0,1,1,1



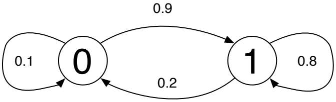
Decoding a unifilar Markov Source



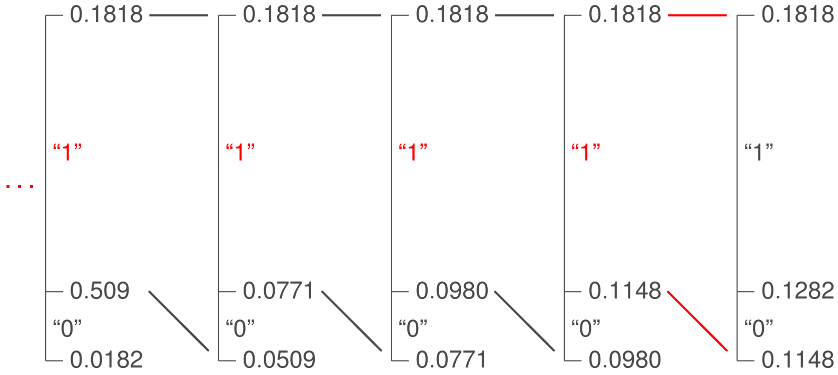
- Decode code sequence: 0,0,1,0,1,0 corresponding to interval $[0.15625, 0.171875]$
- Decoding rule: always pick sub-interval that contains the codeword interval, result: 0,1,1,1,1



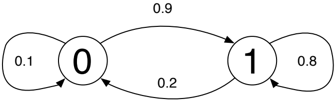
Decoding a unifilar Markov Source



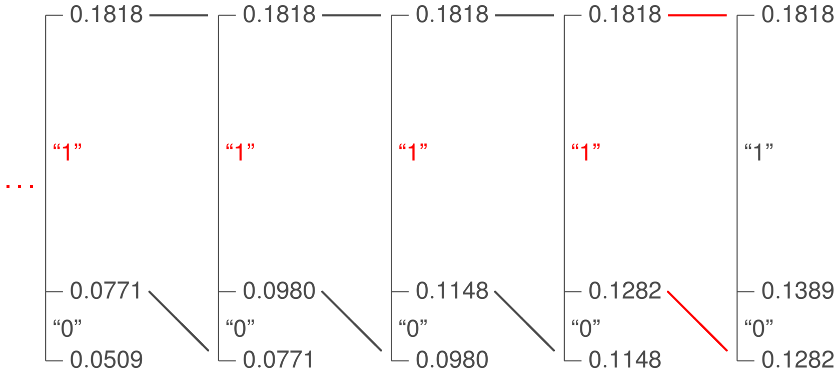
- Decode code sequence: 0,0,1,0,1,0 corresponding to interval $[0.15625, 0.171875]$
- Decoding rule: always pick sub-interval that contains the codeword interval, result: 0,1,1,1,1,1



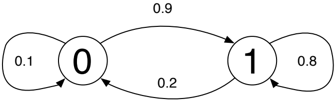
Decoding a unifilar Markov Source



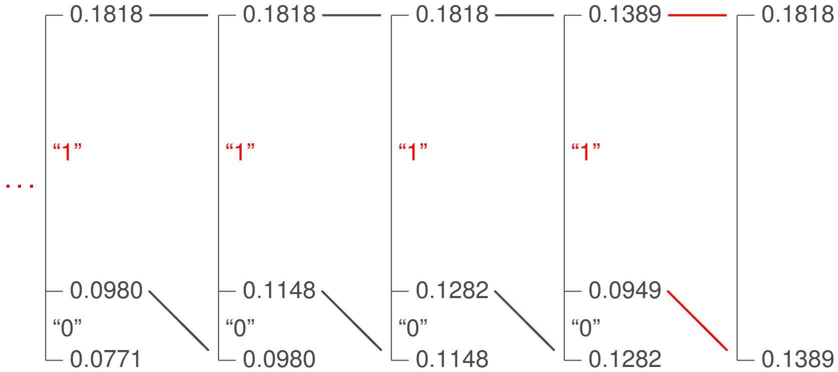
- Decode code sequence: 0,0,1,0,1,0 corresponding to interval $[0.15625, 0.171875]$
- Decoding rule: always pick sub-interval that contains the codeword interval, result: 0,1,1,1,1,1,1



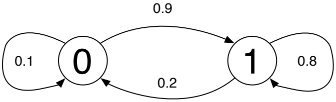
Decoding a unifilar Markov Source



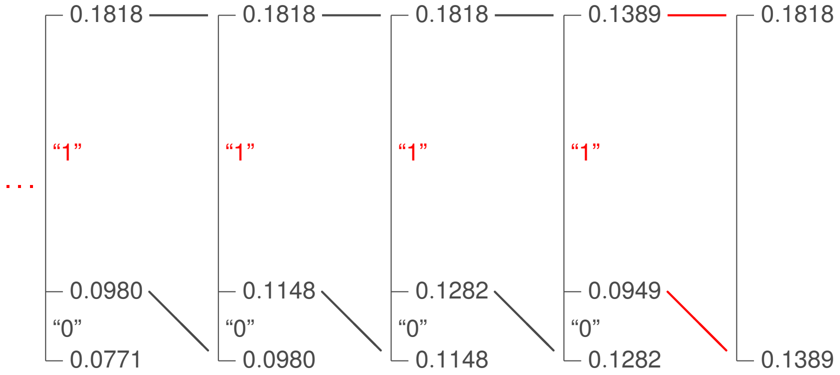
- Decode code sequence: 0,0,1,0,1,0 corresponding to interval $[0.15625, 0.171875]$
- Decoding rule: always pick sub-interval that contains the codeword interval, result: 0,1,1,1,1,1,1,1



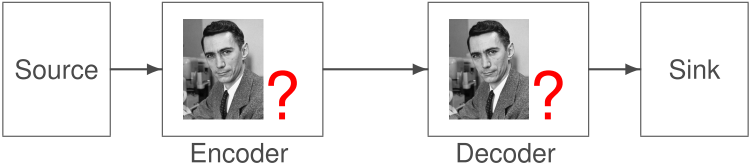
Decoding a unifilar Markov Source



- Decode code sequence: 0,0,1,0,1,0 corresponding to interval $[0.15625, 0.171875]$
- Decoding rule: always pick sub-interval that contains the codeword interval, result: 0,1,1,1,1,1,1,1



Shannon's Twin Experiment



Shannon's twin experiment

- Shannon 1 and Shannon 2 are hypothetical fully identical twins (who look alike, talk alike, and think exactly alike)
- An operator in the transmitter asks Shannon 1 to guess the next source symbol of the source based on the context
- The operator counts the number of guesses until Shannon 1 gets it right, and transmits this number
- An operator in the receiver asks Shannon 2 to guess the next source symbol based on the context. Shannon 2 will get the same answer as Shannon 1 after the same number of guesses.
- An upper bound on the entropy rate of English is the entropy of the number of guesses
- A better bound would take dependencies between numbers of guesses into account (if Shannon 1 needed many guesses for a symbol then chances are that he will need many for the next as well, whereas if he guessed right the first time, chances are that he's in the middle of a word and will guess the next symbol correctly as well)