Engineering Pt IIa, Module 3F3 - Random Processes, Optimal Filtering and Model-based Signal Processing Examples paper

1. Show the following results for jointly wide-sense stationary, discrete-time, real-valued random processes $\{X_n\}$ and $\{Y_n\}$:

   (a) + Autocorrelation function is even:
   $$r_{XX}[k] = r_{XX}[-k]$$

   (b) Cross-correlation function is even
   $$r_{XY}[k] = r_{YX}[-k]$$

   (c) Power spectrum is even:
   $$\mathcal{S}_X(e^{j\Omega}) = \mathcal{S}_X(e^{-j\Omega})$$

   (d) + Power spectrum is periodic:
   $$\mathcal{S}_X(e^{j(\Omega+2n\pi)}) = \mathcal{S}_X(e^{j\Omega})$$

   (e) If $\{Y_n\}$ is the output of a linear system having impulse response $\{h_n\}$ and input $\{X_n\}$, then:
   $$E[Y_n] = E[X_n] \sum_{p=-\infty}^{+\infty} h_p$$
   and
   $$r_{YY}[k] = \sum_{l=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} h_l \, h_i \, r_{XX}(l-i+k) = h_{-k} * h_k * r_{XX}[k]$$
   where '*' denotes discrete time convolution

2. The Bernoulli random process is defined such that each time point is independently assigned a value of +1 or -1 with probabilities p and $1-p$ respectively. A possible sample function from the process is, for example:
   $$\{x_n\} = \{..., -1, +1, +1, -1, +1, +1, ...\}$$

   (a) Determine whether the process is wide-sense stationary

   (b) Is the Bernoulli process a white noise process?

   (c) Determine whether the Bernoulli process is mean ergodic.

3. Determine whether each of the following processes is wide-sense stationary:

   (a) $x_n = R$, where R is a random variable

   (b) $x_n = A\cos(n\Omega_0) + B\sin(n\Omega_0)$ where A and B are uncorrelated random variables with zero mean and identical variances.

   (c) $y_n = x_n - x_{n-1}$, where x_n is the Bernoulli process of the previous question.

4. * The random phase sine wave process is defined as:

$$x_n = A\sin(n\Omega_0 + \phi)$$

where A and Ω_0 are fixed constants and ϕ is a random variable having a uniform probability distribution over the range $-\pi$ to $+\pi$:

$$f(\phi) = \begin{cases} 1/(2\pi) & -\pi < \phi \leq +\pi \\ 0, & \text{otherwise} \end{cases}$$

Determine whether the random-phase sine wave is ergodic in the mean a) when $\Omega_0 = 2m\pi$, b) when $\Omega_0 \neq 2m\pi$, for integer m.

[You may assume the autocorrelation function calculated in the notes:

$$r_{XX}[k] = 0.5A^2\cos(k\Omega_0)$$

]

5. A stationary random process has autocorrelation function:

$$r_{XX}[0] = 2.8, \;\; r_{XX}[\pm 1] = 1$$

and zero values elsewhere. It is known that x_n is a noisy version of another signal d_n:

$$x_n = d_n + v_n$$

where v_n is a zero mean white noise signal with variance equal to 0.5 and uncorrelated with d_n. Determine the optimal second order FIR Wiener filter for estimation of d_n [Hint: first show that $r_{dx}[k] = r_{xx}[k] - r_{vv}[k]$].

Calculate the minimum mean-squared error for this filter.

6. Determine the optimal frequency domain Wiener filter for the same signal as the previous question. Sketch the power spectrum of x_n, d_n, v_n and the frequency response of the filter. Comment on how reasonable the filter seems intuitively.

7. * A stationary AR(1) process can be written as:

$$d_n = a_1 d_{n-1} + e_n$$

where e_n is zero mean white noise with variance σ_e^2.

(a) Determine the power spectrum of the AR(1) process and its autocorrelation function. Sketch the power spectrum of the process for a few values of a_1 between 0 and 1, over the range $\Omega = -4\pi, ..., +4\pi$.

(b) The AR process is now observed in zero mean white noise v_n with variance σ_v^2:

$$x_n = d_n + v_n$$

Assuming that v_n is uncorrelated with x_n, determine the frequency response $H(e^{j\Omega})$ for the IIR Wiener filter which estimates d_n optimally from x_n.

(c) Hence, with $\sigma_v^2 = 0.25$, $\sigma_e^2 = 0.25$ and $a_1 = 0.5$, determine the Wiener filter impulse response. Is this filter causal or non-causal? Sketch the frequency response of this filter and explain why it is intuitively reasonable.

You may use the following DTFT pair:

$$\sum_{n=-\infty}^{\infty} \alpha^{|n|} e^{-jn\Omega} = \frac{1-\alpha^2}{|1-\alpha e^{-j\Omega}|^2}$$

8. * In an aircraft cockpit noise cancellation application a noisy signal is measured at a primary microphone:

$$x_n = d_n + v_{1,n}$$

where d_n is the pilot's voice and $v_{1,n}$ is background noise. At a second microphone in the cockpit a measurement of pure noise $v_{2,n}$ is made. $v_{1,n}$ and $v_{2,n}$ are assumed jointly wide-sense stationary with cross-correlation function $r_{v_1v_2}[k]$, and both are uncorrelated with d_n.

It is required to estimate the voice signal by FIR filtering the noise signal and subtracting from x_n:

$$\hat{d}_n = x_n - \sum_{p=0}^{P} h_p v_{2,n-p}$$

(a) Using the mean-squared error criterion, show that the filter coefficients must satisfy

$$r_{v_2v_1}[k] = \sum_{q=0}^{P} h_q r_{v_2v_2}[k-q]$$

for $k = 0, 1, 2, ..., P$. Hence show that the matrix form of the Wiener-Hopf equations for this case is:

$$\mathbf{R}_{v_2}\mathbf{h} = \mathbf{r}_{v_2v_1}$$

where the matrices and vectors should be carefully defined.

(b) Explain how, in a real environment, $r_{v_2v_2}[k]$ can be measured, assuming v_2 to be ergodic.

(c) Show that when d_n is stationary,

$$r_{v_2v_1}[k] = r_{v_2x}[k]$$

and hence explain how $r_{v_2v_1}[k]$ could be estimated. Is the stationary assumption on d_n realistic in this application?

9. Estimates are made of the correlation function of a particular random process and the values obtained are:

$$r_{XX}[0] = 7.24$$

$$r_{XX}[1] = 3.6$$

Fit a 1st order AR model with transfer function

$$H(z) = \frac{1}{1 + a_1 z^{-1}}$$

to this correlation data. Determine and sketch the resulting power spectrum estimate.

Answers

1.

2. (a) Wide-sense stationary (b) Yes (c) Mean ergodic

3. (a) WSS provided $E[R^2] - E[R]^2$ is finite. (b)WSS. (c)WSS.

4. a) not ergodic b) ergodic

5. $$h_0 = 0.795 \; h_1 = 0.073, \; J_{min} = 0.398$$

6.

7. (a) Power spectrum is
$$\frac{\sigma_e^2}{|(1 - a_1 e^{-j\Omega})|^2}$$
Autocorrelation function is:
$$\frac{\sigma_e^2}{1 - a_1^2} a_1^{|n|}$$
(b)
$$H(e^{j\Omega}) = \frac{\sigma_e^2}{\sigma_e^2 + \sigma_v^2 (1 - a_1 e^{-j\Omega})(1 - a_1 e^{j\Omega})}$$
(c) $h_p = 0.496(0.234)^{|p|}$, non-causal.

8.

9. $\sigma_e = 2.3345, \; a_1 = -0.4972$

Some suitable past tripos questions for both Digital filters and Random processes can be found in the old 4th year module I7, although the style is rather different and the students may find them harder than 3F3 questions.

Some possible ones are: 3F3 2003 - all questions

I7 2002 qqs. 1(final part has a different style from 3F3) 2, 3(a), 4 (but this is harder than typical 3F3)

I7 2001 qqs. 1(a) 2, 3,

I7 2000 qqs. 1, 3, 4

I7 1999 qqs. 1,

I7 1998 qqs. 1,2

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