Question 1

*Signal Space:* Consider the four waveforms \( x_1(t), \ldots, x_4(t) \) shown below.

(a) Determine the dimensionality of the waveforms and a set of orthonormal basis functions.

(b) Use the basis functions to represent the four waveforms by vectors \( x_1, x_2, x_3, x_4 \).

(c) The distance between any two waveforms \( x_i(t), x_j(t) \) can be defined as

\[
d_{ij} = \left( \int (x_i(t) - x_j(t))^2 dt \right)^{\frac{1}{2}}. 
\]

Show that \( d_{ij} = \|x_i - x_j\| \). (Note that \( \|x\| \) denotes the Euclidean norm of the vector \( x \).)

(d) Use part (c) to determine the minimum distance between any pair of waveforms shown above.

Question 2

*Detection with non-uniform symbol probabilities:* Consider BPSK modulation with symbols \( \{+A, -A\} \) over the discrete-time AWGN channel

\[
Y = X + N
\]

where \( N \) is Gaussian noise \( \sim \mathcal{N}(0, N_0/2) \). Suppose that \( P(X = A) = p \) and \( P(X = -A) = 1 - p \).

(a) Derive the detection rule that that minimises the probability of detection error. Sketch the decision regions when \( p = 2/3 \) and \( A/N_0 = 4 \).

(b) Obtain the average probability of detection error, first in terms of \( p, A, N_0 \), then express in terms of \( p \) and \( E_b/N_0 \).
Question 3

M-ary Pulse Amplitude Modulation (PAM): Consider the M-ary PAM constellation shown in the figure below. For \( M \geq 2 \), the constellation consists of \( M \) symbols \( \{p_1, \ldots, p_M\} \) on the real line, symmetric around 0 and with equal spacing \( d \) between symbols. That is,

\[
p_i = (2i - 1 - M)\frac{d}{2}, \quad i = 1, \ldots, M
\]

Suppose that we use this constellation to signal over the discrete-time AWGN channel

\[
Y = X + N
\]

where the Gaussian noise \( N \) is distributed \( \sim \mathcal{N}(0, N_0/2) \). Assuming all the constellation symbols are equally likely:

(a) Sketch the decision regions that minimise the probability of detection error.

(b) Obtain the probability of error when \( p_1 \) or \( p_M \) is sent.

(c) Obtain the probability of error when \( p_i \) is sent, for \( 2 \leq i \leq M - 1 \). Combine this with part (b) to obtain an expression for the overall probability of error \( P_e \).

(d) Show that the average symbol energy \( E_s \) is \( \frac{(M^2 - 1)d^2}{12} \). (Induction may be useful)

(e) Express the probability of error \( P_e \) in terms of \( \frac{E_b}{N_0} \). For fixed \( E_b/N_0 \), how does \( P_e \) vary as \( M \) increases? Is this what you’d expect?

Question 4

Quadrature Phase Shift Keying: Consider QPSK modulation over an AWGN channel

\[
Y = X + N
\]

where the noise \( N \) is a complex random variable distributed as \( \mathcal{CN}(0, N_0) \), i.e., the real and imaginary parts of \( N \) are i.i.d. Gaussian \( \sim \mathcal{N}(0, N_0/2) \). \( X \) is a symbol drawn uniformly from the QPSK constellation below.

\[
p_1 = (-A/\sqrt{2}, A/\sqrt{2}) \quad \quad p_2 = (A/\sqrt{2}, -A/\sqrt{2})
\]

\[
p_3 = (-A/\sqrt{2}, -A/\sqrt{2}) \quad \quad p_4 = (A/\sqrt{2}, A/\sqrt{2})
\]

Sketch the optimal decision regions, and show the probability of detection error \( P_e \leq 2Q(\sqrt{2E_b/N_0}) \).

(The main steps involved in computing \( P_e \) are outlined in Handout 7.)
**Question 5**

*Quadrature Amplitude Modulation*: Consider the 16-QAM constellation shown in the figure below, with adjacent symbols in the vertical and horizontal directions spaced $d$ apart.

![16-QAM constellation diagram]

This constellation is used for signalling (with uniform distribution on the symbols) over the AWGN channel

$$Y = X + N.$$  

The noise $N$ is a complex random variable distributed as $CN(0, N_0)$, i.e., the real and imaginary parts of $N$ are i.i.d. Gaussian $\sim N(0, N_0/2)$.

(a) Derive an upper bound for the probability of error when $X = p_1$ (or $X = p_4/p_{13}/p_{16}$, one of the corner points of the constellation).

(b) Derive an upper bound for the probability of error when $X = p_2$.

(c) Derive an upper bound for the probability of error when $X = p_6$.

(d) Using the union bound show that the average probability of error satisfies

$$P_e \leq 3Q\left(\frac{d}{\sqrt{2N_0}}\right) = 3Q\left(\sqrt{\frac{4E_b}{5N_0}}\right)$$

where $E_b$ is the average energy per bit of the constellation. (For the last equality, you’ll first need to show that $E_s = 2.5d^2$)

**Question 6**

*M-ary FSK*: After demodulation, an $M$-ary FSK receiver has the length-$M$ vector $Y$, given by

$$Y = X + N,$$

where $N_1, \ldots, N_s$ are i.i.d. Gaussian $\sim N(0, N_0/2)$. If message $i$ was transmitted, $X$ has $\sqrt{E_s}$ in the $i$th entry and zeros elsewhere. Note that $E_s = E_b\log_2 M$ is the transmitted energy per symbol.

(a) Derive the optimal detection rule for the $M$-ary FSK receiver.

(b) Show that the probability of detection error can be bounded as $P_e \leq e^{-(\log_2 M)(\frac{E_b}{N_0}-2\ln 2)}$. (The main steps are outlined in Handout 7. You also need to use the bound $Q(x) < \frac{1}{2}e^{-x^2/2}$ for $x > 0$.)

(c) Compare the bandwidth efficiency (rate/bandwidth) of $M$-ary FSK with $M$-ary QAM assuming that the bandwidth of the QAM signal is $2W$ where $W = \frac{1}{T}$. Can you give an intuitive explanation for why QAM is more bandwidth efficient than FSK as $M$ grows large?
(d) How do the probabilities of detection error for the two modulation schemes ($M$-QAM and $M$-FSK) compare as $M$ grows large? (Hint: using a union bound, show that the probability of error for any symbol of square $M$-QAM constellation (like in Q.5) can be bounded by $4Q\left(\frac{d}{\sqrt{2N_0}}\right)$; then use the fact that $d^2 = \kappa E_s = \kappa E_b \log_2 M$ for some constant $\kappa$.)

**Question 7**

*BPSK over a Rayleigh Flat Fading channel*: In Handout 10, we showed that the probability of error for BPSK over a fading channel with coherent detection is given by

$$P_e = \mathbb{E} \left[ Q\left(\sqrt{2|h|^2 \text{snr}}\right) \right] \quad \text{where snr} = \frac{E_b}{N_0}. \quad (1)$$

Recall that $|h|^2$, the squared-magnitude of the fading coefficient $h$ has an exponential density $f$:

$$f(x) = \exp(-x), \quad x \geq 0.$$ 

Show that the average error probability in (1) is equal to $\frac{1}{2} \left(1 - \sqrt{\frac{\text{snr}}{1+\text{snr}}}\right)$.

(Hint: Write the expression in (1) as a double integral and interchange the order of integration.)

**Question 8**

*Diversity via Repetition coding*: Consider the fading channel

$$Y = hX + N$$

In Handout 10, we saw how repetition coding can be used to improve the error performance of BPSK on the fading channel. Here we explore repetition coding with QPSK symbols. Consider $L$ uses of the channel above to transmit a symbol $x$ drawn uniformly from the QPSK constellation shown in Question 6. The output vector is

$$\mathbf{Y} = \mathbf{h}x + \mathbf{N}$$

where $\mathbf{h} = (h[1], \ldots, h[L])^T$ is a vector of complex Gaussian rvs that are i.i.d. $\sim \mathcal{CN}(0,1)$. (We assume that there is interleaving so that the $L$ uses of the channel are over different coherence periods.) $\mathbf{N} = (N[1], N[2], \ldots, N[m])^T$ is a vector of complex Gaussian rvs that are i.i.d. $\sim \mathcal{CN}(0, N_0)$.

We now perform coherent detection.

(a) Project $\mathbf{Y}$ along the direction of $\mathbf{h}$, and observe that the problem reduces to an instance of QPSK detection in AWGN. Write down or sketch the decision regions.

(b) Show that the probability of error conditioned on $\mathbf{h}$ is upper bounded by $2Q\left(\sqrt{\frac{2|h|^2 E_b}{N_0}}\right)$.

(c) The $Q$ function can be upper bounded as $Q(x) < \frac{1}{2} e^{-x^2/2}$ for $x > 0$. Use this to show that

$$P_e|\mathbf{h} \leq \prod_{m=1}^{L} e^{-\frac{E_b}{N_0}|h[m]|^2}$$

(d) Show that the average probability of error is

$$P_e < \left(1 + \frac{E_b}{N_0}\right)^{-L}$$

Hint: Use the fact that the rvs $|h[m]|^2$ for $m = 1, \ldots, L$ are i.i.d. with exponential density $f(x) = e^{-x}, \ x \geq 0$
(e) Repeat the steps above assuming \( x \) came from a 4-PAM constellation \( \{-3d/2, -d/2, d/2, 3d/2\} \). Show that the average probability of error is upper bounded by

\[
P_e < \frac{3}{4} \left(1 + \frac{2E_b}{5N_0}\right)^{-L}.
\]

(Your calculations for Question 5 will be useful)

Thus QPSK has better error performance than 4-PAM as \( E_b/N_0 \) gets large though both transmit 2 bits per symbol. This is because QPSK uses two dimensions, while PAM packs all four symbols along the same dimension.

Question 9

*Diversity via multiple Transmit Antennas:*

Answers

Q1. (d) The minimum distance between waveforms is \( \sqrt{5} \).

Q2. (a) Decode \( \hat{X} = A \) when \( Y \geq T \) and \( \hat{X} = -A \) when \( Y < T \), where the threshold \( T = \frac{N_0}{4A} \ln \left(\frac{1-p}{p}\right) \).

Note that \( T = 0 \), when \( p = \frac{1}{2} \).

(c) \( P_e = p Q \left( \frac{A-T}{\sqrt{N_0/2}} \right) + (1-p) Q \left( \frac{A+T}{\sqrt{N_0/2}} \right) \); \( E_b = A^2 \)

Q3. (b) \( Q \left( \frac{d}{\sqrt{2N_0}} \right) \); (c) \( 2Q \left( \frac{d}{\sqrt{2N_0}} \right) \), overall \( P_e = \frac{2(M-1)}{M} Q \left( \frac{d}{\sqrt{2N_0}} \right) \)

Q5. (a) \( 2Q \left( \frac{d}{\sqrt{2N_0}} \right) \); (b) \( 3Q \left( \frac{d}{\sqrt{2N_0}} \right) \); (c) \( 4Q \left( \frac{d}{\sqrt{2N_0}} \right) \)