4F5: Advanced Communications and Coding

Handout 3: Discrete Channels, Channel Capacity

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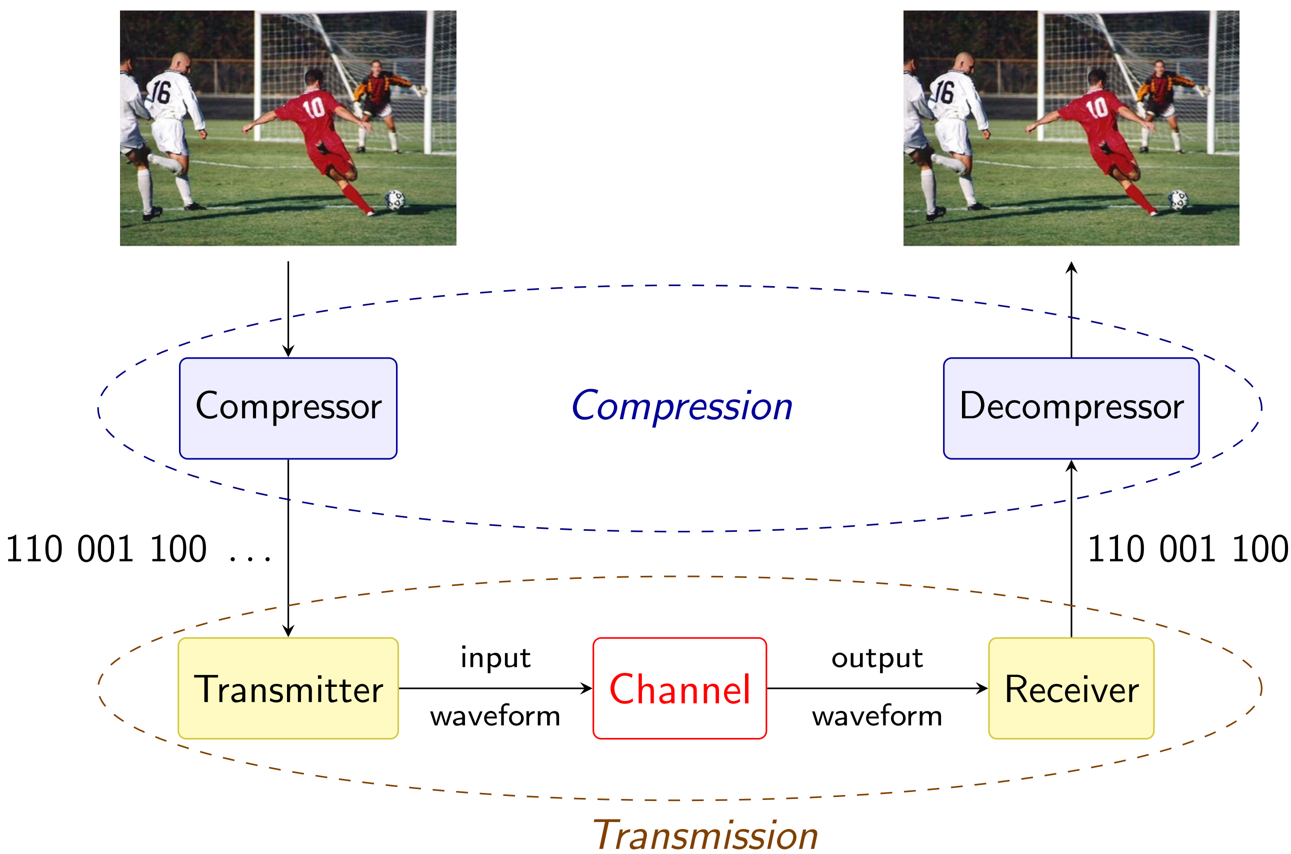
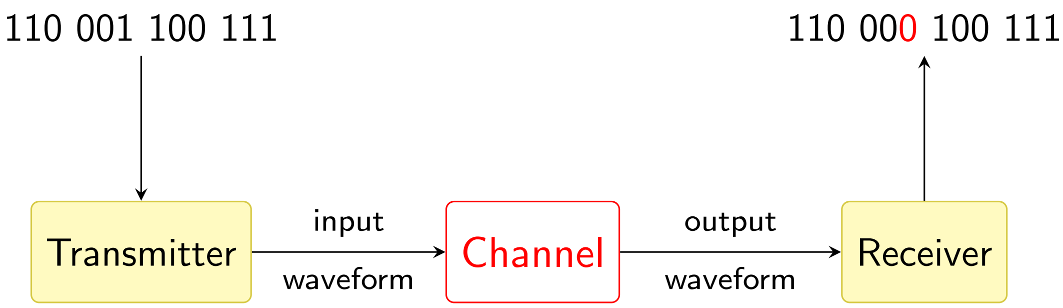
End-to-End Communication System



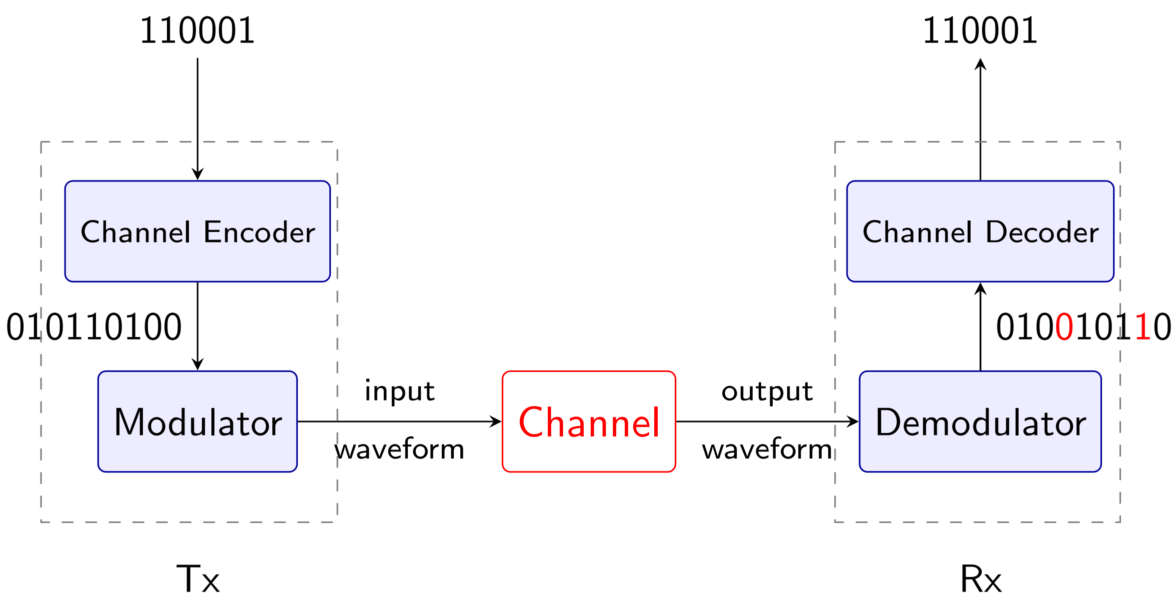
Data Transmission



Transmitter does two things:

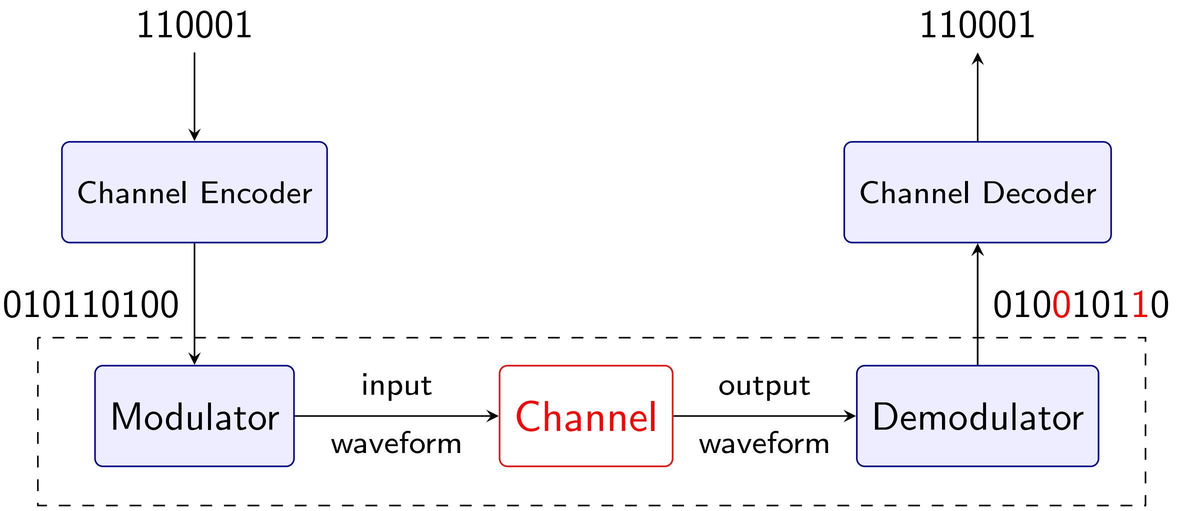
1. *Coding*: Adding redundancy to the data bits to protect against noise
2. *Modulation*: Transforming the coded bits into waveforms. E.g PSK, PAM, QAM etc. are modulation schemes (we'll study these later)

Coding and Modulation



- Optimally, we should design the coding and the modulation in a combined manner.
- However, for engineering simplicity, the modulation scheme and the error-correcting code are often chosen separately.

Binary Channel

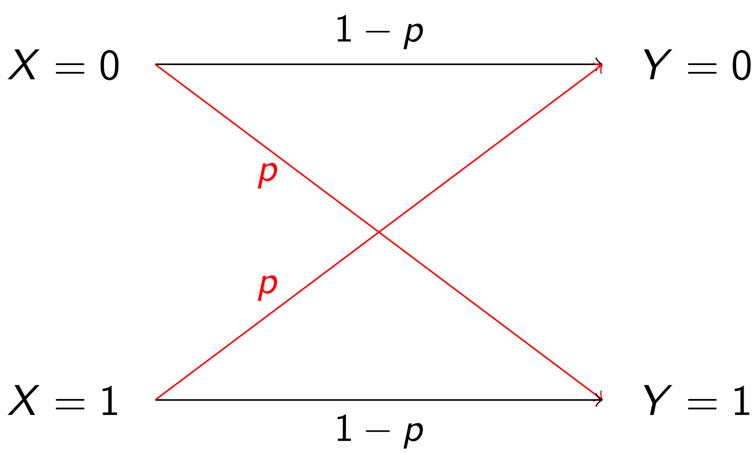


Binary-input, binary-output overall channel with error probability p

- Because of noise in the channel, some bits at the output of the demodulator are in error.
- Every modulation scheme has an associated *probability of error*, say p, that we can estimate theoretically or empirically.

Binary Symmetric Channel (BSC)

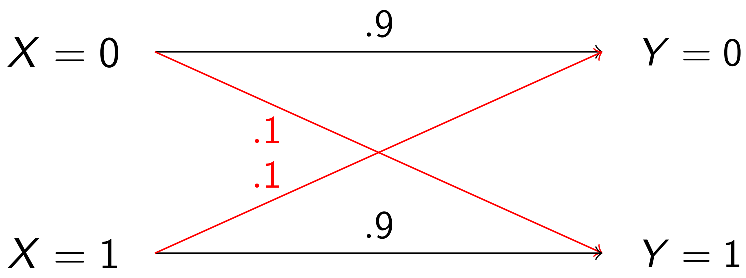
Thus a very important practical channel is the following:



$$P(Y=0|X=0) = 1-p, \quad P(Y=1|X=1) = 1-p$$
$$P(Y=1|X=0) = p, \quad P(Y=0|X=1) = p$$

- p is the "crossover probability"; the channel is called BSC(p)
- KEY Question: How to design good error-correcting codes for the BSC?

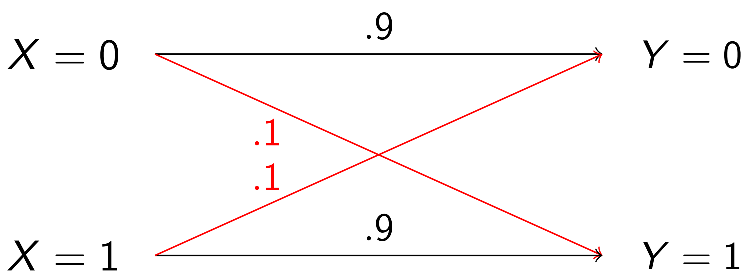
Example: Repetition Code



(1, 3) Repetition Code

Data:	0	1	1	0	0 ...
Coded bits:	000	111	111	000	000 ...
Received bits:	001	101	111	011	000 ...
Decoded bits:	0	1	1	1	0 ...

- Data bit wrongly decoded if channel flips two or more bits in its "codeword"
- Probability of bit decoding error: $\binom{3}{2}(.1)^2(.9) + \binom{3}{3}(.1)^3 = 0.028$
- Data rate $= \frac{1}{3}$ bits/transmission



If we use a $(1,9)$ repetition code:
Probability of bit error $= 0.0009$; Data rate $= \frac{1}{9}$ bits/transmission

As we increase repetition code length n:

- Probability of error goes to 0, but ...
- Data rate $= \frac{1}{n}$, which also goes to 0 ☹

Can we have P(error) $\rightarrow 0$ at strictly positive data rate ?
Yes! [Shannon '48]

For the BSC(0.1) above, we can communicate at rate of as high as 0.53 bit/transmission with *arbitrarily* small probability of error.

Discrete Memoryless Channel

A discrete memoryless channel (DMC) is a system consisting of an input alphabet $\mathcal{X}$, output alphabet $\mathcal{Y}$, and a set of *transition* probabilities

$$P_{Y|X}(b|a) = Pr(Y = b|X = a) \quad \text{for all } a \in \mathcal{X} \text{ and } b \in \mathcal{Y}.$$

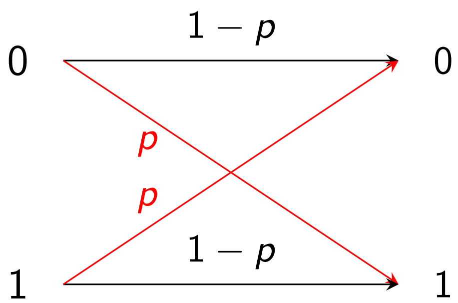
Memoryless means that the channel acts independently on each input. For each time instant $k = 1, 2, \ldots$

$$Pr(Y_k = y|X_k = x, X_{k-1}, \ldots, X_1, Y_{k-1}, \ldots, Y_1) = P_{Y|X}(y|x).$$

"*Given all the past, the current output depends only on the current input*"

DMC Examples

Binary Symmetric Channel (BSC):

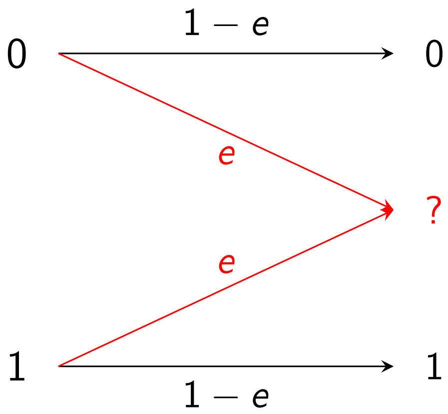


A DMC can be described by a transition probability matrix, e.g.

	$P_{Y\|X}$	Y: 0	Y: 1
X	0	$1-p$	p
	1	p	$1-p$

DMC Examples

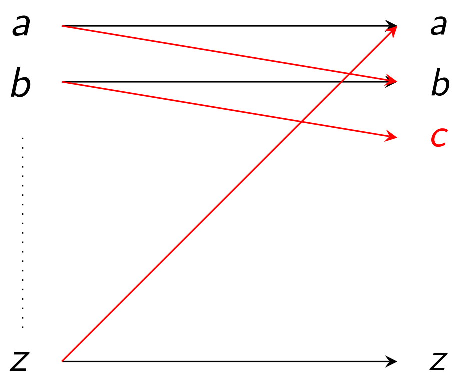
Binary erasure channel (BEC):



		Y		
	$P_{Y \mid X}$	0	?	1
X	0	$1-e$	e	0
	1	0	e	$1-e$

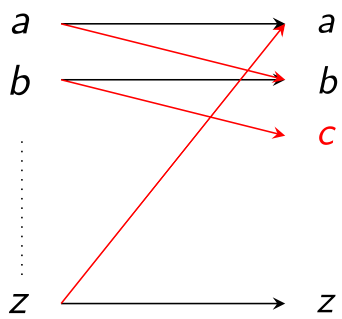
- Like the BSC, the BEC is also practically important, e.g:
  – Erasure can model packet loss in networks
  – When the demodulator thinks the (real-valued) output symbol is too noisy, it can declare an erasure

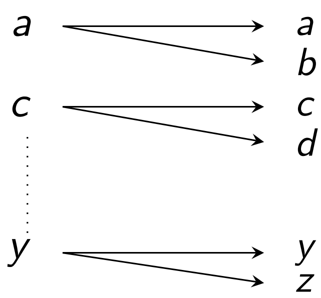
Noisy Keyboard Channel: A useful toy example



- Input alphabet with 26 symbols: $a, b, \ldots, z$
- Each channel input is either received unchanged or transformed into next symbol:

$$P(Y=a|X=a) = P(Y=b|X=a) = \tfrac{1}{2},$$
$$P(Y=b|X=b) = P(Y=c|X=b) = \tfrac{1}{2},$$
$$\vdots$$
$$P(Y=z|X=z) = P(Y=a|X=z) = \tfrac{1}{2}.$$





How can we communicate error-free over this channel?

- A simple code: use only symbols $a, c, e, \ldots, y$
- Can *noiselessly* convey one of 13 symbols per transmission
  $\Rightarrow$ Transmission rate = $\log 13$ bits/transmission
- We will see that this is, in fact, the maximum possible rate

For a general DMC:

- We'll construct a set of input sequences which have *non-intersecting* sets of output sequences with high prob.
- These input sequences are "non-confusable" analogous to inputs $a, c, e, \ldots$ in the keyboard channel.

Channel Capacity

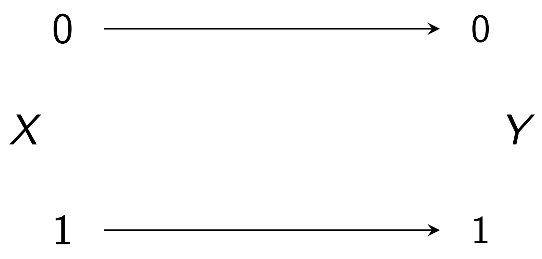
The *channel capacity* of a discrete memoryless channel is defined as

$$\mathcal{C} = \max_{P_X} I(X;Y)$$

- $P_{Y|X}$ is defined by the DMC. Each input distribution P_X yields a different joint distribution on (X, Y) given by $P_X P_{Y|X}$.
- For now, $\mathcal{C}$ is just a mathematical quantity, defined for each DMC.
- We will show that $\mathcal{C}$ is the maximum transmission rate over the DMC if you want arbitrarily small probability of error.

First, let's compute $\mathcal{C}$ for some examples ...

Example 1: Noiseless Binary Channel

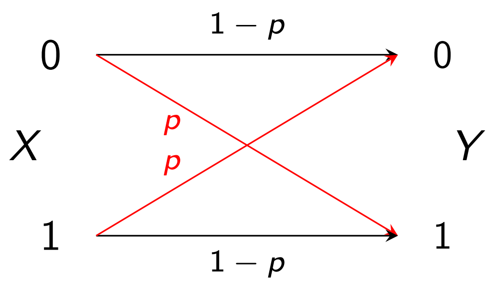


$$I(X;Y) = H(X) - H(X|Y) = H(X) \quad (H(X|Y) = 0)$$

- What P_X maximises $H(X)$? Ans: $P_X = (\frac{1}{2}, \frac{1}{2})$
- Therefore

$$\mathcal{C} = \max_{P_X} I(X;Y) = \max_{P_X} H(X) = 1 \text{ bit/transmission.}$$

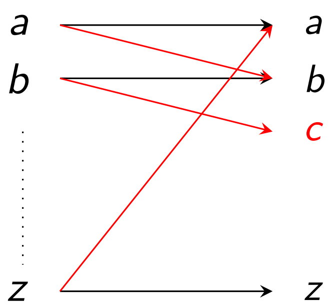
Example 2: BSC



$$\begin{aligned} I(X;Y) &= H(Y) - H(Y|X) \\ &= H(Y) - \sum_{x \in \{0,1\}} P_X(x) H(Y|X=x) \\ &\overset{why?}{=} H(Y) - \sum_{x \in \{0,1\}} P_X(x) H_2(p) \\ &= H(Y) - H_2(p) \\ &\leq 1 - H_2(p). \end{aligned}$$

- Maximum attained when $P_X = (\frac{1}{2}, \frac{1}{2}) \quad \Rightarrow \quad \mathcal{C} = 1 - H_2(p)$
- For $p = 0.1$, $\mathcal{C} = 0.531$ bits/transmission

Example 3: Noisy Keyboard Channel



$$I(X;Y) = H(Y) - H(Y|X)$$
$$= H(Y) - 1 \ \leq \ \log(26) - 1$$

- What P_X maximises $H(Y)$? We have two choices!
  1) $P_X = (1/26,\ 1/26,\ \ldots, 1/26)$
  2) $P_X = (1/13, 0, \ldots, 1/13, 0)$ [We used this one earlier]
- Both yield $P_Y = (1/26,\ 1/26,\ \ldots, 1/26)$

  $$\mathcal{C} = \max_{P_X} I(X;Y) = \log(26) - 1 = \log 13 \text{ bits/transmission.}$$

  MORAL: Maximising input distribution may not be unique!

You can now do Questions 1–13 in Examples Paper 1.