4F5: Advanced Communications and Coding

Handout 5: Data Processing, Fano's Inequality, Channel Coding Converse

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Data Processing and Mutual Information

$$X \longrightarrow \boxed{P_{Y|X}} \longrightarrow Y \longrightarrow \boxed{\text{Processor}} \longrightarrow Z$$

Random variables X, Y, Z are said to form a **Markov chain** if their joint pmf can be written as

$$P_{XYZ} = P_X P_{Y|X} P_{Z|Y}.$$

In other words, the conditional distribution of Z given (X, Y) depends only on Y, i.e., $P_{Z|XY} = P_{Z|Y}$.

Markov chains often occur in engineering problems, e.g.,

1. Y is a noisy version of X, and $Z = f(Y)$ is an estimator of X based only on Y
2. The output of the $X \to Y$ channel is fed into the $Y \to Z$ channel.

Data-Processing Inequality

If X, Y, Z form a Markov chain, then $I(X; Y) \geq I(X; Z)$.

Proof: Q.7, Examples Paper I.

"*Processing the data Y cannot increase the information about X*"

Fano's inequality

$$X \longrightarrow \boxed{P_{Y|X}} \longrightarrow Y \longrightarrow \boxed{\text{Estimator}} \longrightarrow \hat{X} = g(Y)$$

- We want to estimate X by observing a correlated random variable Y
- The probability of error of an estimator $\hat{X} = g(Y)$ is $P_e = \Pr(\hat{X} \neq X)$
- We wish to bound P_e

Fano's Inequality

For *any* estimator $\hat{X}$ such that $X - Y - \hat{X}$, the probability of error $P_e = \Pr(\hat{X} \neq X)$ satisfies

$$1 + P_e \log|\mathcal{X}| \geq H(X|\hat{X}) \geq H(X|Y) \quad \text{or} \quad P_e \geq \frac{H(X|Y) - 1}{\log|\mathcal{X}|}$$

Proof of Fano

- Define an error random variable

$$E = \begin{cases} 1 & \text{if } \hat{X} \neq X \\ 0 & \text{if } \hat{X} = X \end{cases}$$

- Use chain rule to expand $H(E, X|\hat{X})$ in two different ways:

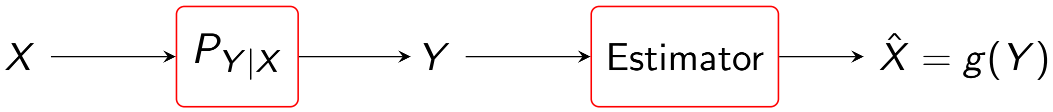
$$\begin{aligned} H(E, X|\hat{X}) &= H(X|\hat{X}) + H(E|X, \hat{X}) \\ &= H(E|\hat{X}) + H(X|\hat{X}, E) \end{aligned} \tag{1}$$

Claims:

1. $H(E|X, \hat{X}) = 0$. (because E is a function of $(X, \hat{X})$)
2. $H(E|\hat{X}) \leq H(E) = H_2(P_e)$. (conditioning can only reduce H)
3. $H(X|\hat{X}, E) \leq P_e \log|\mathcal{X}|$ because

$$\begin{aligned} H(X|\hat{X}, E) &= \Pr(E = 0) H(X|\hat{X}, E = 0) + \Pr(E = 1) H(X|\hat{X}, E = 1) \\ &\leq (1 - P_e)\, 0 + P_e \log|\mathcal{X}| \end{aligned}$$

Using the three claims in (1), we get ...



$$H(X|\hat{X}) \leq H_2(P_e) + P_e \log|\mathcal{X}|$$

Note that $H_2(P_e) \leq 1$. Therefore

$$H(X|\hat{X}) \leq 1 + P_e \log|\mathcal{X}|.$$

We have proved one side of Fano.

For the other side, the data-processing inequality tells us that

$$I(X;Y) = H(X) - H(X|Y) \geq I(X;\hat{X}) = H(X) - H(X|\hat{X})$$

Thus $H(X|\hat{X}) \geq H(X|Y)$. □

Back to the Channel Coding problem . . .

Fano's Inequality applied to a channel code:

- Consider a $(2^{nR}, n)$ channel code
- $\hat{W}$ is a guess of W based on Y^n
- W uniformly distributed in $\{1, \ldots, 2^{nR}\}$
- $P_e = \Pr(\hat{W} \neq W) = \frac{1}{2^{nR}} \sum_{k=1}^{2^{nR}} \Pr(\hat{W} \neq k | W = k)$

Fano's inequality applied to this problem gives:

$$H(W|\hat{W}) \leq 1 + P_e \log 2^{nR} = 1 + P_e\, nR$$

We will use this to show that *any* sequence of $(2^{nR}, n)$ codes with $P_e \to 0$ must have $R \leq \mathcal{C}$.

A Little Lemma

Let Y^n be the result of passing a sequence X^n through a DMC of channel capacity $\mathcal{C}$. Then

$$I(X^n; Y^n) \leq n\mathcal{C}$$

regardless of the distribution of X^n.

$$\begin{aligned}
\textit{Proof}: \quad I(X^n; Y^n) &= H(Y^n) - H(Y^n|X^n) \\
&= H(Y^n) - \sum_{i=1}^{n} H(Y_i|Y_{i-1}, \ldots, Y_1, X^n) \\
&\overset{(a)}{=} H(Y^n) - \sum_{i=1}^{n} H(Y_i|X_i) \\
&\overset{(b)}{\leq} \sum_{i=1}^{n} H(Y_i) - \sum_{i=1}^{n} H(Y_i|X_i) \\
&= \sum_{i=1}^{n} I(X_i; Y_i) \overset{(c)}{\leq} n\mathcal{C}.
\end{aligned}$$

Justification for steps $(a) - (c)$:

(a) The channel is assumed to be *memoryless*. This means that given X_i, Y_i is conditionally independent of everything else.

(b) We have

$$\begin{aligned}
H(Y^n) &= H(Y_1) + H(Y_2|Y_1) + \ldots + H(Y_n|Y_{n-1}, \ldots, Y_1) \\
&\leq H(Y_1) + H(Y_2) + \ldots + H(Y_n)
\end{aligned}$$

as conditioning can only reduce entropy.

(c) From the definition of capacity, $\mathcal{C}$ is the maximum of $I(X; Y)$ over all joint pmfs over (X, Y) where $P_{Y|X}$ is fixed by the channel.

The Converse (Part 2 of the Channel Coding Theorem)

Consider *any* $(2^{nR}, n)$ channel code with average probability of error P_e. We have:

$$\begin{aligned} nR &\overset{(a)}{=} H(W) \\ &\overset{(b)}{=} H(W|\hat{W}) + I(W;\hat{W}) \\ &\overset{(c)}{\leq} 1 + P_e nR + I(W;\hat{W}) \\ &\overset{(d)}{\leq} 1 + P_e nR + I(X^n;Y^n) \\ &\overset{(e)}{\leq} 1 + P_e nR + n\mathcal{C}. \end{aligned}$$

This implies:

$$P_e \geq 1 - \frac{\mathcal{C}}{R} - \frac{1}{nR}$$

Thus, unless $R \leq \mathcal{C}$, P_e is bounded *away* from 0 as $n \to \infty$. $\square$

Justification for steps $(a) - (e)$:

(a) W is uniform over $\{1, \ldots, 2^{nR}\}$

(b) $I(W;\hat{W}) = H(W) - H(W|\hat{W})$

(c) Fano applied to $H(W|\hat{W})$ (see Slide 6)

(d) Data processing inequality applied to $W - X^n - Y^n - \hat{W}$.

(e) From the lemma on Slide 7

Summary

$\mathcal{C}$ is a sharp threshold!

- For all rates $R < \mathcal{C}$, there exists a sequence of $(2^{nR}, n)$ codes whose $P_e \to 0$.
- For $R > \mathcal{C}$, you cannot find a sequence of $(2^{nR}, n)$ codes whose $P_e \to 0$.

Given a channel, do we have a practical way to communicate reliably at any rate $R < \mathcal{C}$?

No, because

1. Joint typical decoding is too complex to be feasible
2. An $2^{nR} \times n$ codebook too large to store

In the next six lectures (by Jossy), you will learn how to design good channel codes with

- Compact codebook representation
- Fast encoding and decoding algorithms

You can now do all the questions in Examples Paper 1