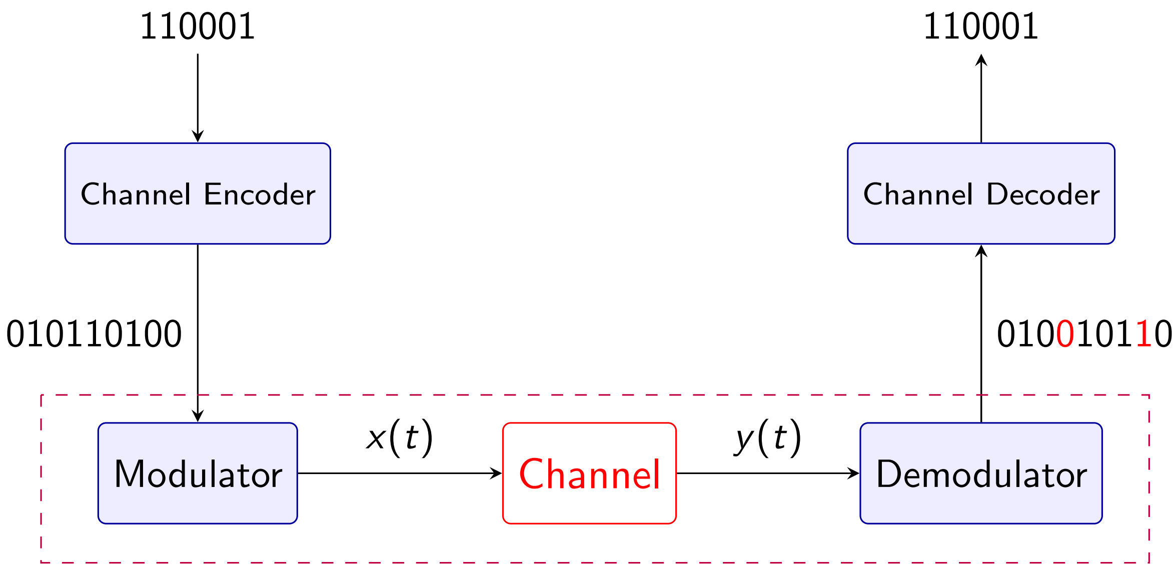
4F5: Advanced Communications and Coding

Handout 6: Signal Space and Modulation

Ramji Venkataramanan

Signal Processing and Communications Lab
Department of Engineering
ramji.v@eng.cam.ac.uk

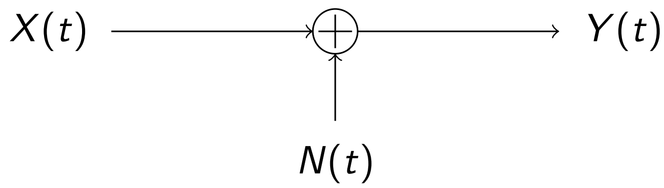
Michaelmas Term 2015



In the last five lectures of this module:

- We will look at some real-world channel models and learn to design modulators & demodulators for these channels
- We begin with the classic additive white Gaussian noise (AWGN) channel, and then study wireless channels, which have “fading” in addition to additive noise

The Additive White Gaussian Noise (AWGN) Channel



The *continuous-time* AWGN channel:

$$Y(t) = X(t) + N(t)$$

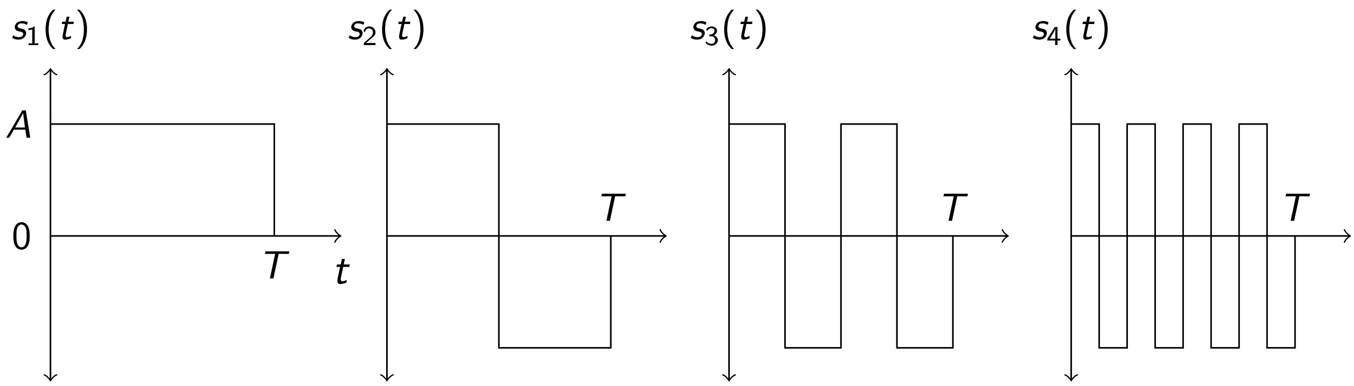
1. Input $X(t)$ is *power-limited* to P:
   $\Rightarrow$ Average energy over time T must be $\leq P$ for large T
2. $X(t)$ is *band-limited* $\Rightarrow$
   Fourier Transform of $X(t)$ must be zero outside the interval $[f_c - W, f_c + W]$, where f_c is the carrier frequency and $f_c \gg W$
3. Noise $N(t)$ is a random process assumed to be *white* Gaussian
   This means that for each t, $N(t)$ is a Gaussian rv with

$$\mathbb{E}[N(t)] = 0, \quad \mathbb{E}[N(t)N(s)] = \frac{N_0}{2}\delta(t-s), \qquad \text{for all } t, s$$

Mapping bits to waveforms

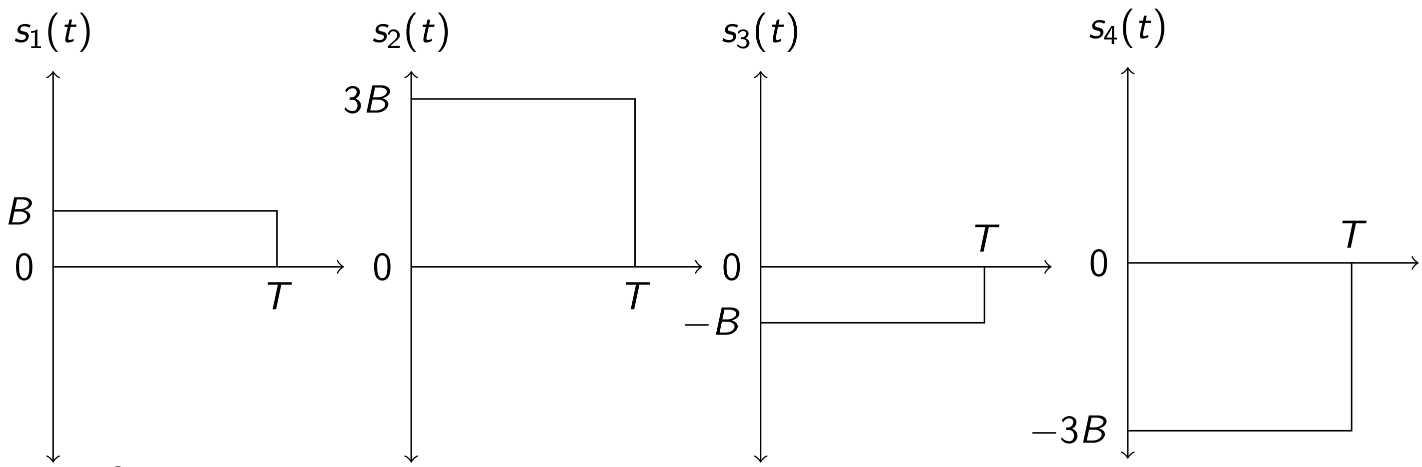
- Suppose we want to transmit m bits in T seconds using a waveform $x(t),\ t \in [0, T)$.
- Note that each m-bit pattern indexes one of $M = 2^m$ messages
- We choose a collection of M waveforms $\{s_1(t), \ldots, s_M(t)\}$. Map each m-bit sequence to one of these waveforms.

Example: $m = 2$ bits. One choice of $M = 4$ waveforms could be

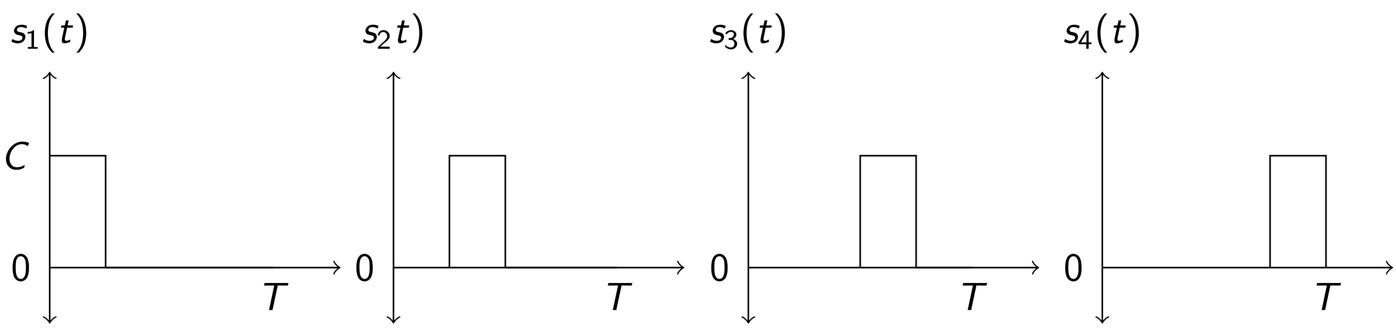


$00 \to s_1(t)$ $\quad$ $01 \to s_2(t)$ $\quad$ $10 \to s_3(t)$ $\quad$ $11 \to s_4(t)$

We could also choose something like:



OR



- We need a systematic way to represent finite-energy functions defined in an interval $[0, T]$.
- We will see that by fixing a set of basis functions, signal waveforms can be represented as "vectors", which obey properties similar to vectors in Euclidean space $\mathbb{R}^n$.
- Let us first review what a vector space is ...

Vector Spaces

A vector space $\mathcal{V}$ is a set of elements (called "vectors") that is **closed under addition and scalar multiplication**. The scalars are often chosen to be real or complex-valued, but they can be chosen over some other field as well.

That is, if $\underline{v}_1, \underline{v}_2 \in \mathcal{V}$, then $a\underline{v}_1 + b\underline{v}_2 \in \mathcal{V}$, for any scalars a, b.

You are already familiar with the Euclidean vector space $\mathbb{R}^k$.

A set of *linearly independent* vectors, say $\{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k\}$, is called a **basis** of the vector space $\mathcal{V}$, if $v \in \mathcal{V}$ can be expressed as a linear combination of the form

$$\underline{v} = a_1\,\underline{v}_1 + a_2\,\underline{v}_2 + \ldots + a_k\,\underline{v}_k$$

for some scalars $a_1, \ldots, a_k$.
If the number of vectors in the basis is a finite number k, the vector space is said to have *dimension* k.
Q: Specify a basis for $\mathbb{R}^3$. Is the basis unique?

Inner product, Orthogonality, Norm

You are familiar with the *inner-product* ("dot-product") in Euclidean space $\mathbb{R}^n$: if

$$\underline{v}_1 = x_1\underline{e}_1 + x_2\underline{e}_2 + \ldots + x_n\underline{e}_n,$$
$$\underline{v}_2 = y_1\underline{e}_1 + y_2\underline{e}_2 + \ldots + y_n\underline{e}_n,$$

then

$$< \underline{v}_1, \underline{v}_2 >= x_1 y_1 + \ldots + x_n y_n.$$

(Here $\underline{e}_1, \ldots, \underline{e}_n$ are the unit vectors along the n axes of $\mathbb{R}^n$)

- Recall that vectors $\underline{v}_1$ and $\underline{v}_2$ are *orthogonal* if $< \underline{v}_1, \underline{v}_2 >= 0$. E.g. in $\mathbb{R}^n$, $\underline{e}_i, \underline{e}_j$ are orthogonal iff $i \neq j$.
- The **norm** ("length") of a vector $\underline{v}$ is defined as

$$\|\underline{v}\| = \sqrt{< \underline{v}, \underline{v} >}$$

We will now extend these concepts to a vector space of *signals*. This vector space is called the *signal space*.

The Signal Space

To understand communication over the continuous-time channel

$$Y(t) = X(t) + N(t),$$

it is useful to consider the vector-space of *finite-energy* signals. Let $\mathcal{L}_2$ be the set of complex-valued signals (functions) $x(t)$ with finite energy, i.e.,

$$\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$$

It can be shown that $\mathcal{L}_2$ is a vector space (set of finite-energy signals is closed under addition and scalar multiplication.)

- The inner product in this space can be defined as follows. For $x(.), y(.) \in \mathcal{L}_2$,

$$< x, y > = \int_{-\infty}^{\infty} x(t) y^*(t) dt$$

- The norm of a signal is the square-root of its energy:

$$\|x\| = \sqrt{< x, x >} = \left[\int_{-\infty}^{\infty} |x(t)|^2 dt \right]^{1/2}$$

Orthonormal Basis

For any vector space $\mathcal{L} \subset \mathcal{L}_2$, the set of functions $\{f_i(.),\ i = 1, 2, \ldots\}$ is called an **orthonormal** basis for $\mathcal{L}$ if

1. Every $x(.) \in \mathcal{L}$ can be expressed as

$$x(t) = \sum_i x_i f_i(t),$$

   for some *scalars* $x_1, x_2, \ldots$, and

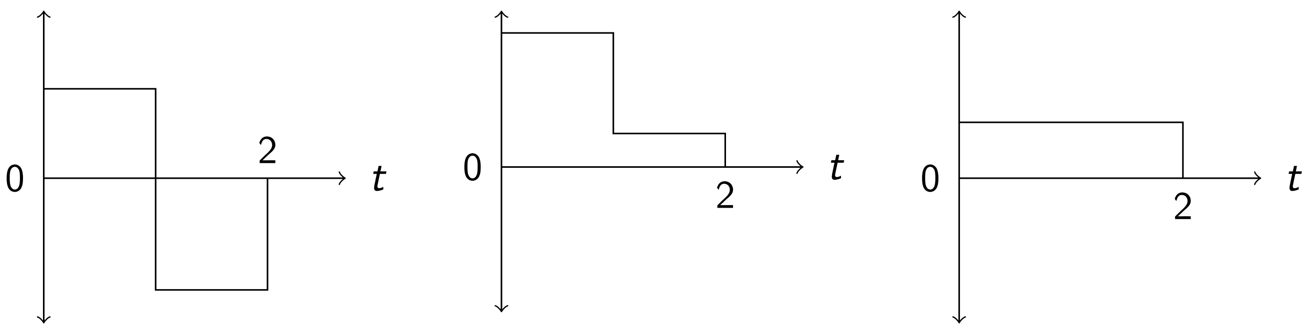
2. The functions $\{f_i(.),\ i = 1, 2, \ldots\}$ are *orthonormal*, i.e.,

$$< f_\ell, f_m > = \int f_\ell(t) f_m^*(t) dt = \begin{cases} 1 & if\ \ell = m \\ 0 & if\ \ell \neq m \end{cases}$$

If these two conditions are satisfied, the orthonormal basis $\{f_i\}$ is said to *span* the vector space $\mathcal{L}$.
The number of elements (functions) in the basis is called the *dimension* of the vector space $\mathcal{L}$

Example

Let $\mathcal{L}$ be the set of all functions $f(t)$ defined for $t \in [0, 2]$ that are piece-wise constant in the intervals $[0, 1]$ and $(1, 2]$.
E.g., some such functions are:



- This set of functions is a vector space (why?)
- An orthonormal basis for this space is:

What is the dimension of this vector space?

Exercise: Find an orthonormal basis for the vector space spanned by each of the three sets of functions in p.5–6. Also specify the dimension of the space in each case.

Why do we care about an orthonormal basis for a given vector space $\mathcal{L}$? Consider any function $x(.) \in \mathcal{L}$, expressed in terms of an orthonormal basis $\{f_i(t)\}$ as

$$x(t) = \sum_i x_i f_i(t).$$

Each coefficient x_j can be calculated as

$$x_j = < x, f_j > = \int \Big(\sum_i x_i f_i(t) \Big) f_j^*(t) dt = \sum_i x_i \int f_i(t) f_j^*(t) dt = x_j$$

The coefficients $x_1, x_2, \ldots$ are called the *projections* of the signal $x(t)$ along $f_1(t), f_2(t), \ldots$, respectively.

The inner product between $x(t)$ and $y(t)$ is

$$< x(t), y(t) > = \int \Big(\sum_i x_i f_i(t) \Big) \Big(\sum_j y_j f_j(t) \Big)^* dt = \sum_i x_i y_i^*$$

The energy of $x(t)$ can thus be written as

$$\int |x(t)|^2 dt = < x(t), x(t) > = \sum_i |x_i|^2$$

Thus, if we fix an orthonormal basis for $\mathcal{L}$, we can treat functions in $\mathcal{L}$ just like vectors in Euclidean space, i.e.,

$$x(t) \leftrightarrow (x_1, x_2, \dots x_k),$$

where k is the number of elements in the orthonormal basis, i.e., the *dimension* of the signal space.

- We can use this vector representation to easily compute inner products between signals, energies etc.
- We have effectively converted continuous-time operations (integrals over t) into discrete-time operations (summations)!

Signal space is a convenient framework to analyse modulation and demodulation.

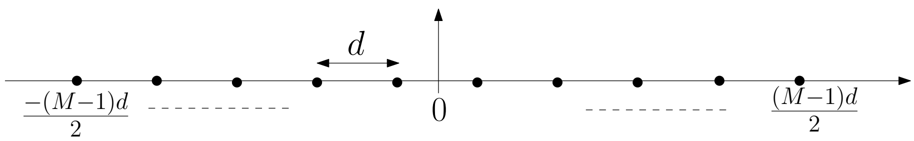
Modulation Techniques

We now describe some common modulation techniques in terms of their associated signal space over one symbol period $[0, T)$.

M-ary Pulse Amplitude Modulation (PAM):

$$x(t) = x \cdot p(t), \quad t \in [0, T)$$

1. $p(t)$ is a unit energy pulse, e.g., $p(t) = \frac{1}{\sqrt{T}}$ for $t \in [0, T)$
2. The symbol x can take one of M values, chosen from the set $\left\{ -(2i - 1 - M)\frac{d}{2} \right\}$, $i = 1, \dots, M$



$x(t)$ lies in a 1-dim. signal space with basis $f_1(t) = p(t)$. Hence

$$x(t) \longleftrightarrow x$$

where x belongs to the above constellation.

Quadrature Amplitude Modulation (QAM)

In M-ary QAM, the symbol x is chosen from a *complex* constellation with M symbols. Over one symbol period:

$$x(t) = \mathrm{Re}(x)p(t)\sqrt{2}\cos(2\pi f_c t) + \mathrm{Im}(x)p(t)\sqrt{2}\sin(2\pi f_c t), \quad t \in [0, T)$$

- As in PAM, $p(t)$ is a baseband unit energy pulse, whose spectrum is (almost) zero for frequencies outside $[-W, W]$
- The carrier frequency $f_c \gg W \approx \frac{1}{2T}$. For example, $f_c = 1$ GHz, $\frac{1}{T} = 1$ MHz
- The real part of the symbol modulates the cosine, the imaginary part modulates the sine.

$x(t)$ lies in 2-dim. signal space with orthonormal basis functions

$$f_1(t) = \sqrt{2}p(t)\cos(2\pi f_c t), \quad f_2(t) = \sqrt{2}p(t)\sin(2\pi f_c t),$$

Therefore $x(t) = \sum_{k=1}^{2} x_k f_k(t)$ where $[x_1, x_2] = [\mathrm{Re}(x), \mathrm{Im}(x)]$, where x belongs to the constellation

The Orthonormal Basis for QAM

We want to show that

$$\int_0^T f_1^2(t)dt = 1, \quad \int_0^T f_2^2(t)dt = 1, \quad \int_0^T f_1(t)f_2(t)dt = 0$$

We have

$$\int_0^T f_1^2(t)dt = \int_0^T 2p^2(t)\cos^2(2\pi f_c t)dt = \int_0^T p^2(t)(1+\cos(4\pi f_c t))dt$$

$$= 1 + \int_0^T p^2(t)\cos(4\pi f_c t)dt$$

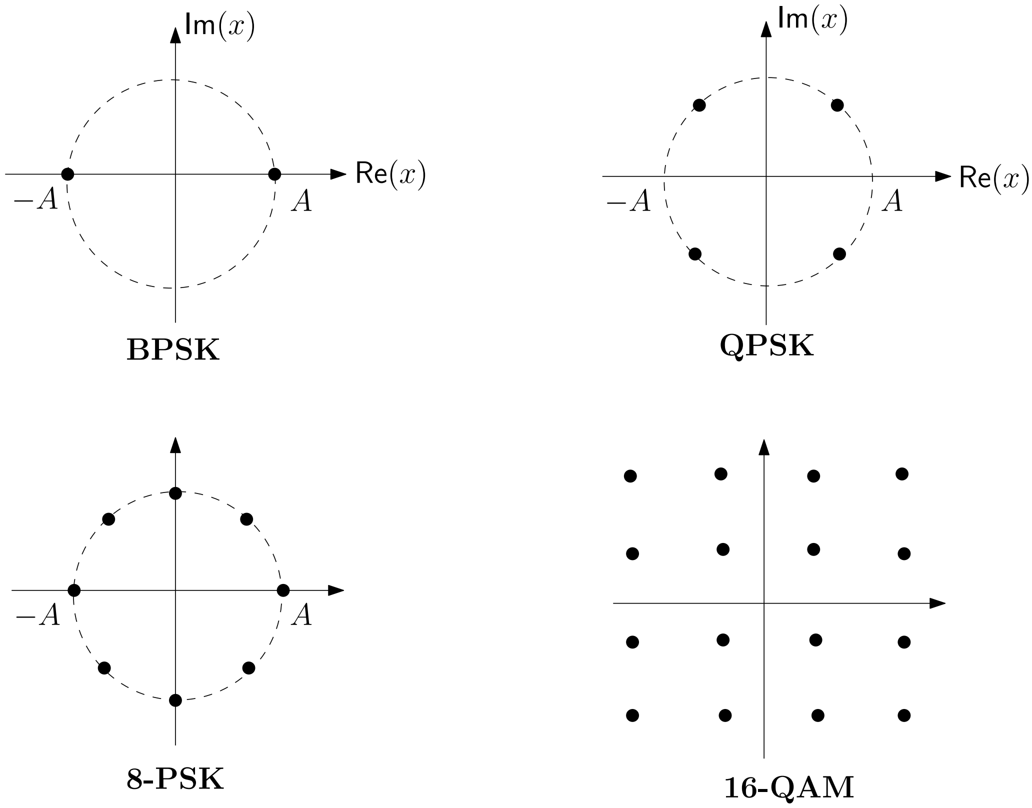
Note $\cos(4\pi f_c t)$ has period $\frac{1}{2f_c} \ll T$. Assume $T = \frac{K}{2f_c}$ for some integer K. So we can split the above integral as

$$\int_0^{\frac{1}{2f_c}} p^2(t)\cos(4\pi f_c t)dt + \int_{\frac{1}{2f_c}}^{\frac{2}{2f_c}} p^2(t)\cos(4\pi f_c t)dt \ldots + \int_{\frac{(K-1)}{2f_c}}^{\frac{K}{2f_c}} p^2(t)\cos(4\pi f_c t)dt \tag{1}$$

As $p(t)$ is a baseband signal, it can be assumed to be nearly constant over a period of $\frac{1}{2f_c}$. Therefore each of the K integrals ≈ 0, and hence $\int f_1^2(t) \approx 1$. Similar calculations show that $\int f_2^2(t) \approx 1$ and $\int f_1(t)f_2(t)dt \approx 0$.

Some QAM Constellations

$$x_{QAM}(t) = \text{Re}(x)p(t)\sqrt{2}\cos(2\pi f_c t) + \text{Im}(x)p(t)\sqrt{2}\sin(2\pi f_c t),\ t \in [0, T)$$



PAM and QAM are digital versions of amplitude modulation as the information symbols modulate the amplitude of the pulse/carrier.

We now study at a modulation scheme where the information symbols modulate the *frequency* of the carrier.

Orthogonal Signalling (M-ary FSK)

To send message $i \in \{1, \ldots, M\}$ in the symbol period $[0, T)$:

$$x(t) = \sqrt{\frac{2E_s}{T}} \cos\left(2\pi\left(f_c + (2i - (M+1))\frac{\Delta_f}{2}\right) t\right),$$

where $\Delta_f = \frac{1}{2T} = \frac{f_c}{K}$ for some large integer K

- The M symbols are represented by M cosines, whose frequencies are separated by multiples of Δ_f:

$$f_c - \frac{(M-1)}{2}\Delta_f, \quad f_c - \frac{(M-3)}{2}\Delta_f, \ldots, \quad f_c + \frac{(M-1)}{2}\Delta_f$$

- We have an M-**dimensional** signal space with ortho-basis: $f_i(t) = \left\{\sqrt{\frac{2}{T}} \cos\left(2\pi\left(f_c + (2i-(M+1))\frac{\Delta_f}{2}\right) t\right)\right\}$, $t \in [0, T)$, $\quad i = 1, \ldots, M$.
- Hence $x(t) = \sum_{k=1}^{M} x_k f_k(t)$. If the ith message is transmitted, the only non-zero term in the sum is $x_i = \sqrt{E_s}$, i.e.,

$$[x_1, \ldots, x_i, \ldots x_M] = [0, \ldots, \sqrt{E_s}, \ldots, 0]$$

- E_s is the energy of $x(t)$ in one period $[0, T)$

Orthonormal Basis for M-ary FSK

We want to show that $\int_0^T f_i^2(t)dt = 1$ for all $i \in \{1, \ldots, M\}$, and $\int_0^T f_i(t) f_j(t) dt = 0$ for $i \neq j$.

$$\int_0^T f_i^2(t)dt = \frac{1}{T}\int_0^T \left[1 + \cos\left(4\pi\left(f_c + (2i-(M+1))\frac{\Delta_f}{2}\right) t\right)\right] dt$$

$$= 1 - \frac{1}{cT}\underbrace{\sin\left(4\pi\left(f_c + (2i-(M+1))\frac{\Delta_f}{2}\right) T\right)}_{=\ 0 \text{ as } f_cT = \frac{K}{2} \text{ and } \Delta_f T = \frac{1}{2}}$$

Similarly, for $i \neq j$

$$\int_0^T f_i(t) f_j(t) dt = \frac{1}{T}\int_0^T \Big[\cos\left(2\pi\left(2f_c + ((i+j)-(M+1))\Delta_f\right) t\right) - \cos\left(2\pi\ (i-j)\ \Delta_f t\right)\Big] dt$$

$$= \frac{1}{cT}\underbrace{\sin\left(2\pi\left(2f_c + ((i+j)-(M+1))\Delta_f\right) T\right)}_{0} - \frac{1}{c'T}\underbrace{\sin\left(2\pi\ (i-j)\ \Delta_f T\right)}_{0}$$

(c, c' are constants whose exact values don't matter)

Demodulation (without noise)

To summarize, we considered modulation schemes of the form:

Message $\in \{1, \ldots, M\} \quad \longleftrightarrow \quad [x_1, \ldots, x_K] \quad \longrightarrow \quad \sum_{i=1}^{K} x_i f_i(t)$

where

- $\{f_i(t)\}_{i=1}^{K}$ is an set of functions which form an orthonormal basis for the signal space of $x(t)$.
- $K = 1$ for PAM, $K = 2$ for QAM, $K = M$ for M-ary FSK.

Demodulation is the process of recovering the message — or, equivalently $[x_1, \ldots, x_K]$ — from the received waveform $y(t)$.

- Assuming there is no noise, $y(t) = x(t)$
- To recover x_j, compute inner product of (x) with $f_j(t)$:

$$< x, f_j >= \int \Big(\sum_i x_i f_i(t) \Big) f_j^*(t) dt = \sum_i x_i \int f_i(t) f_j^*(t) dt = x_j$$

- We can recover $[x_1, \ldots, x_K]$ (and hence the message) by just computing inner products of $x(t)$ with $f_1(t), \ldots, f_K(t)$.
- In the next handout, we'll study how to perform demodulation in the presence of noise.