4F5: Advanced Communications and Coding

Handout 7: Demodulation & Detection in AWGN noise

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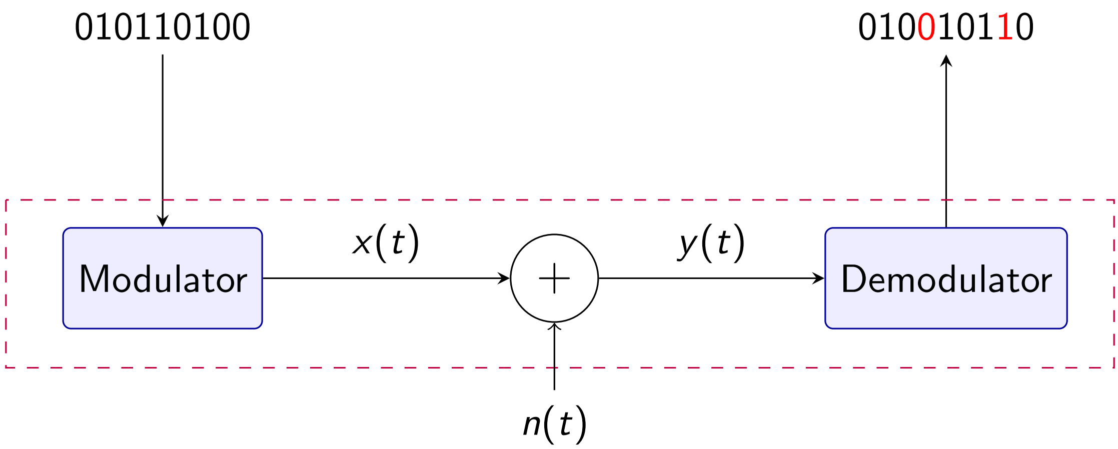


- Having fixed the modulation scheme, we know the K-dim. orthogonal basis for the signal space of transmitted waveform
- Modulator converts:

$$\text{Bits} \longrightarrow K\text{-dim. vector} \longrightarrow \text{Transmitted waveform}$$

- Demodulator converts:

$$\text{Received waveform} \longrightarrow K\text{-dim. vector} \longrightarrow \text{Bits}$$

- In this handout, we'll study QAM and FSK demodulation in the presence of AWGN noise.

QAM Modulation

(1) In each time period of length T, we transmit one QAM symbol drawn from a QAM constellation. A mapping from bits to complex points. E.g.

BPSK: $0 \to A, \quad 1 \to -A$

QPSK:

$$\begin{aligned} 00 &\to \tfrac{A}{\sqrt{2}}(1+j), & 01 &\to \tfrac{A}{\sqrt{2}}(1-j), \\ 10 &\to \tfrac{A}{\sqrt{2}}(-1+j), & 11 &\to \tfrac{A}{\sqrt{2}}(-1-j) \end{aligned}$$

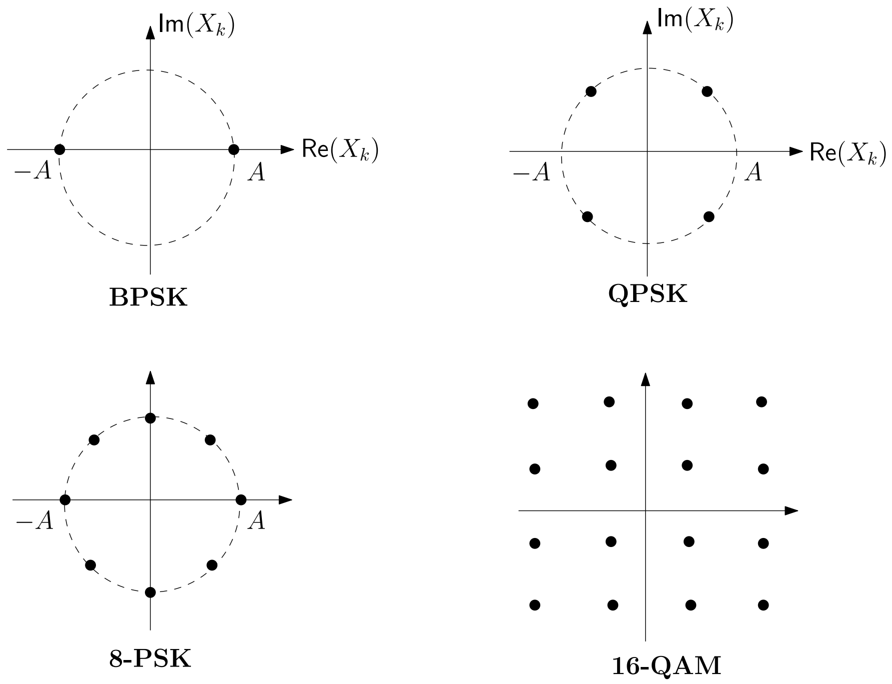
(2) To transmit symbol X_k ($k = 0, 1, \ldots$), the QAM waveform is:

$$\mathrm{Re}(X_k)p(t-kT)\sqrt{2}\cos(2\pi f_c t) + \mathrm{Im}(X_k)p(t-kT)\sqrt{2}\sin(2\pi f_c t)$$

(3) The final QAM waveform is

$$\begin{aligned} X(t) = \sum_k \Big[&\mathrm{Re}(X_k)p(t-kT)\sqrt{2}\cos(2\pi f_c t) \\ &+ \mathrm{Im}(X_k)p(t-kT)\sqrt{2}\sin(2\pi f_c t) \Big] \end{aligned}$$

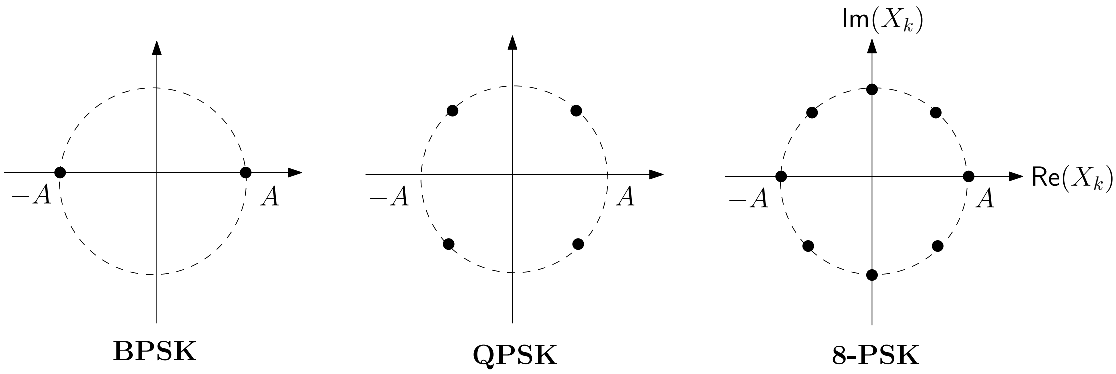
Some Common Signal Constellations



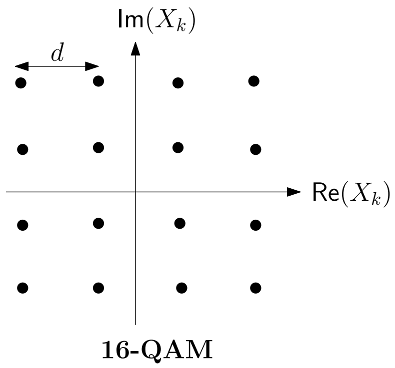
In "Phase Shift Keying" (PSK), the magnitude of X_k is constant, and the information is contained in the phase of the symbol.

In an M-symbol constellation, each symbol corresponds to $\log_2 \mathbf{M}$ bits

Average Energy per Symbol



For all the PSK constellations, average *symbol* energy $E_s = A^2$



Average energy per symbol for 16-QAM

$$E_s = \frac{40d^2}{16} = 2.5d^2$$

Average energy per *bit* $E_b = E_s / \log M$

How to choose the Pulse $p(t)$?

$p(t)$ is chosen to satisfy the following important objectives:

1. We want $p(t)$ to decay quickly in time, i.e., the effect of *symbol* X_k should not start much before $t = kT$ or last much beyond $t = (k+1)T$
2. We want $p(t)$ to be approximately band-limited to $[-W, W]$. For a fixed sequence of symbols $\{X_k\}$, the QAM waveform $X(t)$ will then be band-limited to $[f_c + W, f_c - W]$. E.g., $f_c = 2.4$GHz , $W = 1$ MHz
3. The retrieval of the information sequence from the *noisy* received waveform should be simple and relatively reliable. In the absence of noise, the symbols $\{X_k\}_{k\in\mathbb{Z}}$ should be recovered perfectly at the receiver.

Orthonormality of pulse shifts

Consider the third objective, namely, simple and reliable detection.

To achieve this, the pulse is chosen to have the following "**orthonormal shift**" property:

$$\int_{-\infty}^{\infty} p(t-kT)p(t-mT)\,dt = \begin{cases} 1 & \text{if } k=m \\ 0 & \text{if } k \neq m \end{cases} \tag{1}$$

We'll later see how this property helps demodulation at the Rx.

Let's first look at some candidates for $p(t)$ ...

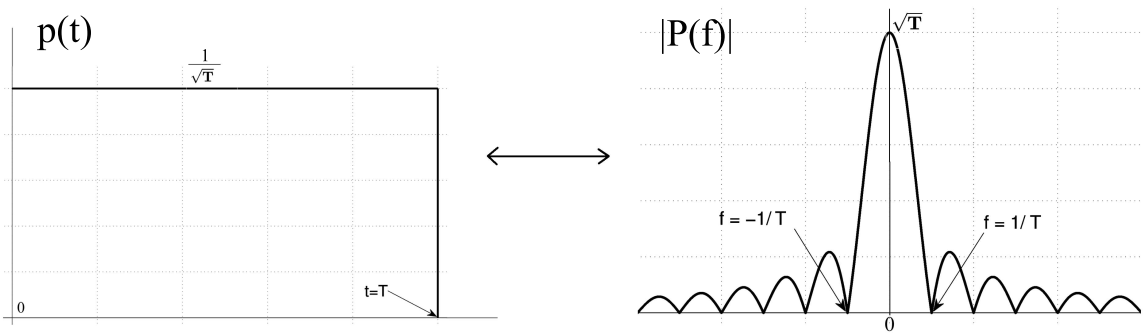
Time Decay vs. Bandwidth Trade-off

The first two objectives say that we want $p(t)$ to:

1. Decay quickly in time
2. Be approximately band-limited

But ... faster decay in time $\Leftrightarrow$ larger bandwidth

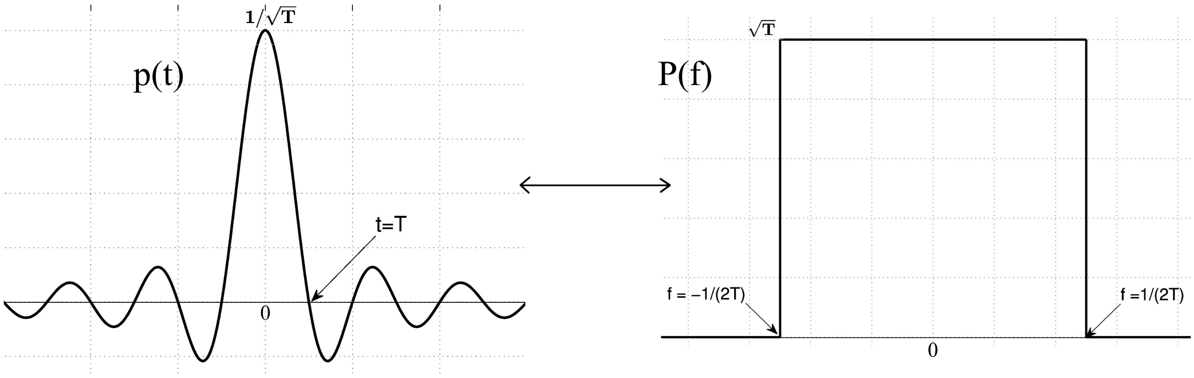
(1) Consider the rectangular pulse $p(t) = \begin{cases} \frac{1}{\sqrt{T}} & \text{for } t \in (0, T] \\ 0 & \text{otherwise} \end{cases}$



The rect. is perfectly time-limited with the symbol interval $[0, T)$

But decays slowly in freq. $|P(f)| \sim \frac{1}{|f|}$; main-lobe bandwidth $= \frac{1}{T}$

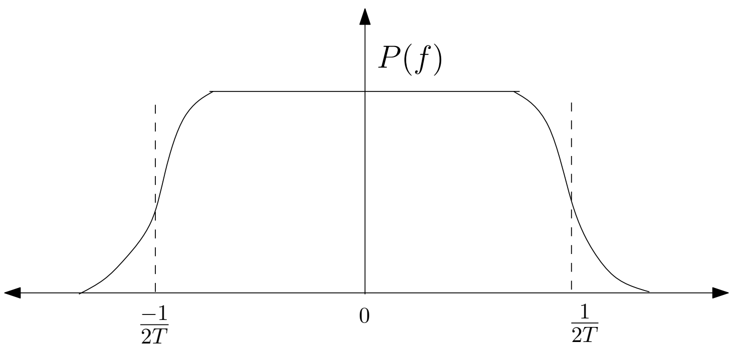
(2) Next consider the pulse $p(t) = \frac{1}{\sqrt{T}} \operatorname{sinc}\left(\frac{\pi t}{T}\right)$



The sinc is perfectly band-limited to $W = \frac{1}{2T}$

But decays slowly in time $|p(t)| \sim \frac{1}{|t|}$

(3) In practice, the pulse shape is often chosen to have a *raised cosine* spectrum:



Bandwidth slightly larger than $\frac{1}{2T}$; decay in time $|p(t)| \sim \frac{1}{|t|^3}$

A happy compromise!

The rect, sinc, and raised cosine all satisfy the orthogonal shifts property:

$$\int p(t - kT)p(t - mT)\, dt = \begin{cases} 1 & \text{if } k = m \\ 0 & \text{if } k \neq m \end{cases}$$

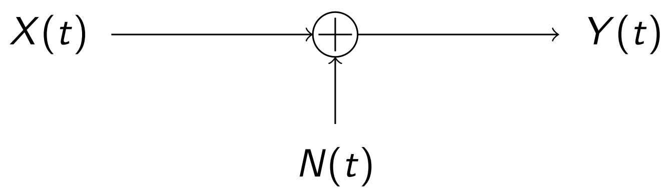
The Received Waveform $Y(t)$

Recall that the transmitted QAM waveform is

$$X(t) = \sum_k p(t - kT)\left[X_k^r \sqrt{2}\cos(2\pi f_c t) + X_k^i \sqrt{2}\sin(2\pi f_c t)\right]$$

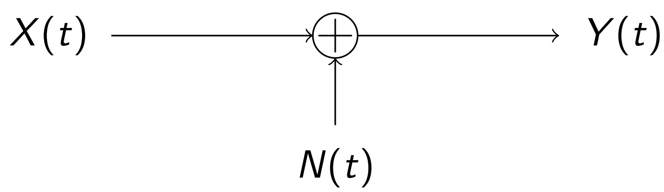
where $X^r = \mathrm{Re}(X_k)$, and $X^i = \mathrm{Im}(X_k)$

The *transmission rate* is $\frac{1}{T}$ **symbols/sec** or $\frac{\log_2 M}{T}$ **bits/second**



- The receiver gets $Y(t) = X(t) + N(t)$
- $N(t)$ is a white Gaussian process with

$$\mathbb{E}[N(t)] = 0, \quad \mathbb{E}[N(t)N(s)] = \frac{N_0}{2}\delta(t-s)$$



The received waveform is

$$\begin{aligned} Y(t) &= X(t) + N(t) \\ &= \sum_k X_k^r\, f_k(t)\sqrt{2}\cos(2\pi f_c t) + X_k^i\, f_k(t)\sqrt{2}\sin(2\pi f_c t) + N(t) \end{aligned}$$

where $f_k(t) = p(t - kT)$

Key Fact

The functions $\left\{ f_\ell(t)\sqrt{2}\cos(2\pi f_c t), \quad f_\ell(t)\sqrt{2}\sin(2\pi f_c t)\right\}, \ \ell \in \mathbb{Z}$ form an *orthonormal* set.

Proof of Key Fact:

$$\int_{-\infty}^{\infty} f_\ell(t) f_m(t) 2\cos^2(2\pi f_c t) dt = \int f_\ell(t) f_m(t)(1 + \cos(4\pi f_c t)) dt$$
$$\overset{(a)}{\approx} 1 \text{ if } \ell = m, \quad 0 \text{ if } l \neq m$$

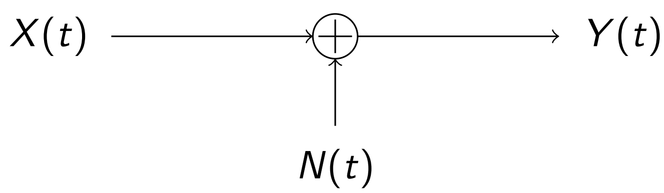
(a) holds because $f_c \gg W \geq \frac{1}{2T}$, i.e., $\cos(4\pi f_c t)$ completes many, many cycles in the time taken for $f_\ell(t), f_m(t)$ to change appreciably. (This is similar to the proof in Handout 6, p.18)

Similarly

$$\int_{-\infty}^{\infty} f_\ell(t) f_m(t) 2\sin^2(2\pi f_c t) dt = \int f_\ell(t) f_m(t)(1 - \cos(4\pi f_c t)) dt$$
$$\approx 1 \text{ if } l = m, \quad 0 \text{ if } \ell \neq m$$

$$\int_{-\infty}^{\infty} f_\ell(t) f_m(t) 2\sin(2\pi f_c t)\cos(2\pi f_c t) dt = \int f_\ell(t) f_m(t) \sin(4\pi f_c t)) dt$$
$$\approx 0 \text{ for all } \ell, m \quad \square$$

At the Receiver



The receiver performs three steps:

1. *Demodulation*: Convert the received waveform $Y(t)$ into a discrete-time sequence $Y_1, Y_2, \ldots$ by projecting $Y(t)$ onto the elements of the orthonormal set
$\{f_k(t)\sqrt{2}\cos(2\pi f_c t), \quad f_k(t)\sqrt{2}\sin(2\pi f_c t)\}, \ k \in \mathbb{Z}$
2. *Detection*: Recover $\hat{X}_1, \hat{X}_2, \ldots$ from $Y_1, Y_2, \ldots$
($\hat{X}_1, \hat{X}_2, \ldots$ are points in the constellation)
3. Convert $\hat{X}_1, \hat{X}_2, \ldots$ to bits using the assignment of bits to constellation points (easy)

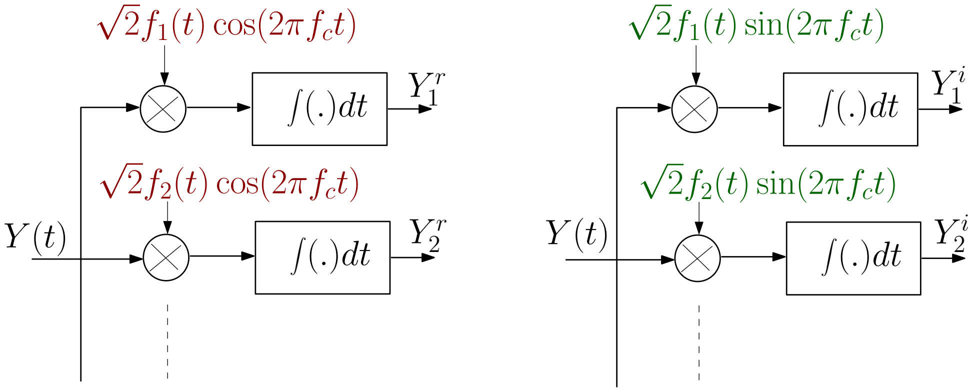
Step 1: QAM Demodulation

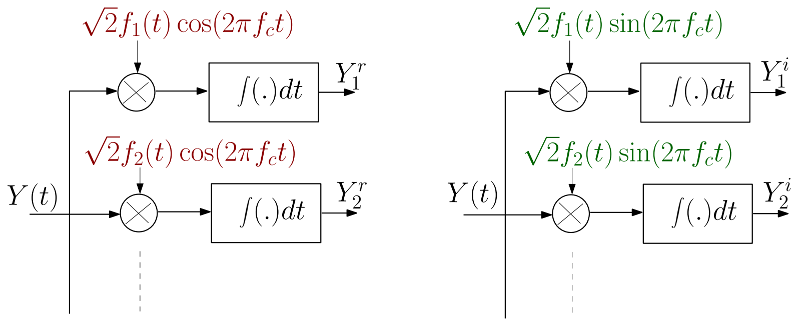
$$Y(t) = \sum_k X_k^r \sqrt{2} f_k(t) \cos(2\pi f_c t) + X_k^i \sqrt{2} f_k(t) \sin(2\pi f_c t) + N(t)$$

We generate for each ℓ: $Y_\ell^r = < Y(t),\ f_\ell(t)\sqrt{2}\cos(2\pi f_c t) >$

$$Y_\ell^i = < Y(t),\ f_\ell(t)\sqrt{2}\sin(2\pi f_c t) >$$

Implemented via a bank of correlators (or using "matched filters")





Since the functions $\left\{f_k(t)\sqrt{2}\cos(2\pi f_c t),\quad f_k(t)\sqrt{2}\sin(2\pi f_c t)\right\}$ are orthonormal, the outputs of the demodulator (the bank of correlators) for $k \in \mathbb{Z}$ are:

$$Y_k^r = < Y(t),\ \sqrt{2}f_k(t)\cos(2\pi f_c t) > = X_k^r + N_k^r$$
$$Y_k^i = < Y(t),\ \sqrt{2}f_k(t)\sin(2\pi f_c t) > = X_k^i + N_k^i$$

where

$$N_k^r = < N(t), \sqrt{2}f_k(t)\cos(2\pi f_c t) >,\ N_k^i = < N(t), \sqrt{2}f_k(t)\sin(2\pi f_c t) >$$

- The next step is *detection*: how to recover to the information symbols $\{X_k^r, X_k^i\}$ from $\{Y_k^r, Y_k^i\}$?
- For this we first need to understand the statistics of $\{N_k^r, N_k^i\}$

For any orthonormal set of functions $\{\phi_k(t)\}_{k\in\mathbb{Z}}$, let us compute the joint distribution of $\{N_k\}$, where $N_k = \int_{-\infty}^{\infty} N(t)\phi_k(t)dt$

For each k, N_k is a linear combination of *jointly* Gaussian rvs $\{N(t),\ t \in \mathbb{R}\} \Rightarrow$ The rvs $\{N_k\}$, for $k \in \mathbb{Z}$ are jointly Gaussian

1. For each integer k:

$$\mathbb{E}[N_k] = \mathbb{E}\Big[\int_{-\infty}^{\infty} N(t)\phi_k(t)dt\Big] = \int_{-\infty}^{\infty} \mathbb{E}[N(t)]\phi_k(t)dt = 0$$

2. For each pair of integers k, ℓ:

$$\mathbb{E}[N_k N_\ell] = \mathbb{E}\Big[\int_{-\infty}^{\infty} N(t)\phi_k(t)dt \int_{-\infty}^{\infty} N(s)\phi_\ell(s)dt\Big]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbb{E}[N(t)N(s)]\ \phi_k(t)\phi_\ell(s)dt\ ds$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{N_0}{2}\delta(t-s)\ \phi_k(t)\phi_\ell(s)dt\ ds$$

$$= \frac{N_0}{2}\int_{-\infty}^{\infty} \phi_k(s)\phi_\ell(s)ds = \frac{N_0}{2} \text{ if } k = \ell \text{ and } 0 \text{ otherwise}$$

We apply this result to the orthonormal set

$$\{\sqrt{2}f_k(t)\cos(2\pi f_c t),\ \sqrt{2}f_k(t)\sin(2\pi f_c t)\}_{k\in\mathbb{Z}}$$

$$N_k^r = \int_{-\infty}^{\infty} N(t)\,\sqrt{2}f_k(t)\cos(2\pi f_c t)dt$$

$$N_k^i = \int_{-\infty}^{\infty} N(t)\,\sqrt{2}f_k(t)\sin(2\pi f_c t)dt$$

The rvs $\{N_k^r, N_k^i\},\ k \in \mathbb{Z}$ are jointly Gaussian with:

$$\mathbb{E}[N_k^r] = \mathbb{E}[N_k^i] = 0$$

$$\mathbb{E}[N_k^r N_\ell^r] = \frac{N_0}{2} \text{ if } k = \ell \text{ and } 0 \text{ otherwise}$$

$$\mathbb{E}[N_k^i N_\ell^i] = \frac{N_0}{2} \text{ if } k = \ell \text{ and } 0 \text{ otherwise}$$

$$\mathbb{E}[N_k^i N_\ell^r] = 0 \text{ for all } k, \ell$$

Therefore the collection of rvs $\{N_k^r\}$ and $\{N_k^i\}$ for $k \in \mathbb{Z}$ are i.i.d Gaussian with each distributed as $\mathcal{N}(0, \frac{N_0}{2})$

Step 2: QAM Detection

The receiver gets $\mathbf{Y}_k = \mathbf{X}_k + \mathbf{N}_k, \quad$ for each $k = 1, 2, \ldots$

[$\mathbf{Y}$ denotes (Y^r, Y^i), $\mathbf{X} = (X^r, X^i)$ and $\mathbf{N} = (N^r, N^i)$]

Detection

For each k, how to recover $\mathbf{X}_k$ from $\mathbf{Y}_k$?

Recall that N_k^r, N_k^i are i.i.d $\mathcal{N}(0, \frac{N_0}{2})$ for all k. Hence

$$P(\mathbf{Y} = (y^r, y^i)|\mathbf{X} = (x^r, x^i)) = \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(y^r - x^r)^2}{N_0}} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(y^i - x^i)^2}{N_0}}$$

The Maximum A Posteriori (MAP) Decoder

The decoding rule that minimises the probability of error is

$$\hat{\mathbf{X}} = \arg\max_{\mathbf{x}} \; P(\mathbf{X} = \mathbf{x}|\mathbf{Y} = \mathbf{y})$$

where the arg max is over all $\mathbf{x}$ in the *constellation*

The MAP Decoding Rule

$$\hat{\mathbf{X}} = \arg\max_{\mathbf{x}} \; P(\mathbf{Y} = \mathbf{y}|\mathbf{X} = \mathbf{x}) \; P(\mathbf{X} = \mathbf{x})$$

If all the symbols in the constellation are equally likely, i.e., $P(\mathbf{x})$ is the same for all symbols), then

$$\begin{aligned}
\hat{\mathbf{X}} &= \arg\max_{\mathbf{x}} \; P(\mathbf{Y} = \mathbf{y}|\mathbf{X} = \mathbf{x}) \\
&= \arg\max_{\mathbf{x}} \frac{1}{\pi N_0} \; e^{-\frac{(y^r - x^r)^2}{N_0}} \; e^{-\frac{(y^i - x^i)^2}{N_0}} \\
&= \arg\max_{\mathbf{x}} e^{-\frac{1}{N_0}\|\mathbf{y}-\mathbf{x}\|^2} \\
&= \arg\min_{\mathbf{x}} \; \|\mathbf{y} - \mathbf{x}\|^2
\end{aligned}$$

If all the constellation symbols are equally likely, the optimum detector simply chooses the symbol *closest* to the output.
(Also called 'nearest-neighbour' or 'maximum-likelihood' decoding)

MAP Decoding Examples

Note that

$$\|\mathbf{y}-\mathbf{x}\|^2 = \|\mathbf{y}\|^2 + \|\mathbf{x}\|^2 - 2\mathbf{x}\mathbf{y}^T = \|\mathbf{y}\|^2 + \|\mathbf{x}\|^2 - 2(x^r y^r + x^i y^i)$$

The term $\|\mathbf{y}\|^2$ does not affect detection. Therefore, the MAP decoding rule for equally likely symbols becomes

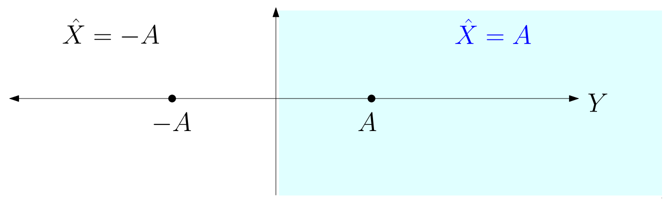
$$\hat{\mathbf{X}} = \arg\min_{\mathbf{x}} \|\mathbf{x}\|^2 - 2\mathbf{x}\mathbf{y}^T$$

1) BPSK: $\mathbf{X} \in \{A, -A\}$

$$\hat{\mathbf{X}} = \arg\min \|\mathbf{y}-\mathbf{x}\|^2 = \arg\max \; x^r y$$

where the arg max is over $x^r \in \{A, -A\}$.
The MAP *decision regions* are

$$\hat{\mathbf{X}} = A \text{ if } y > 0, \;\; \hat{\mathbf{X}} = -A \text{ if } y < 0$$



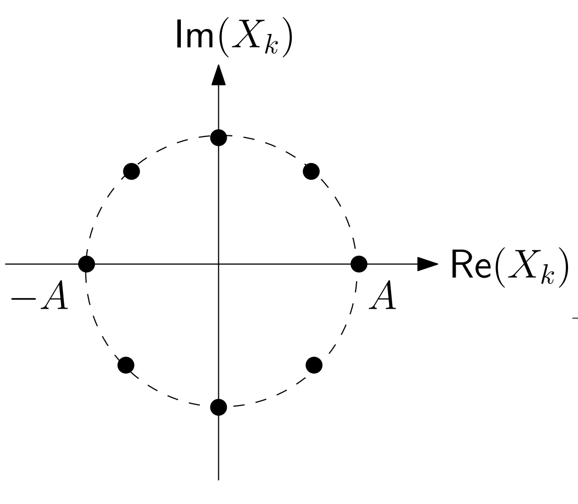
MAP Decoding Examples ctd.

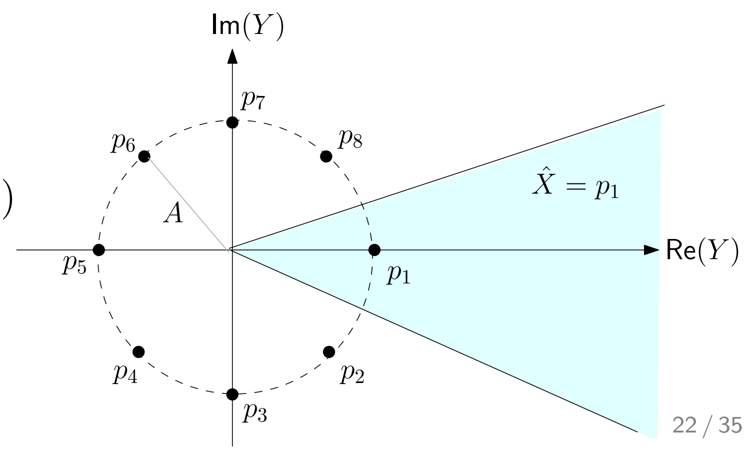
2) 8-PSK: Let $\theta(\mathbf{x}, \mathbf{y})$ be the angle between vectors $(\mathbf{x}, \mathbf{y})$.

$$\|\mathbf{y}-\mathbf{x}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2\|\mathbf{x}\|\|\mathbf{y}\|\cos(\theta(\mathbf{x}, \mathbf{y}))$$

As $\|\mathbf{x}\| = A$ for all symbols in 8-PSK, the MAP decoding rule is:

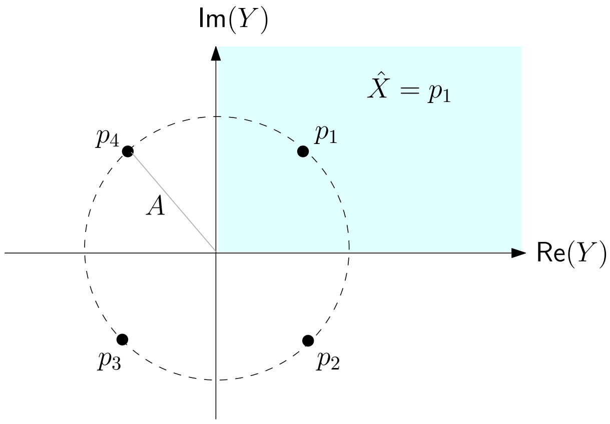
$$\hat{\mathbf{X}} = \arg\min_{\mathbf{x}} \; \theta(\mathbf{x}, \mathbf{y})$$



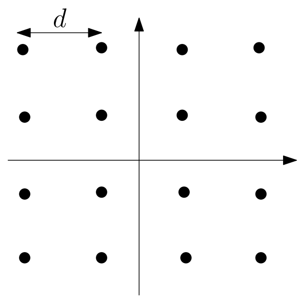


3) QPSK: Similar min-angle decoding as $8 - PSK$.

$$\hat{\mathbf{X}} = \arg\min_{\mathbf{x}} \ \theta(\mathbf{x}, \mathbf{y})$$

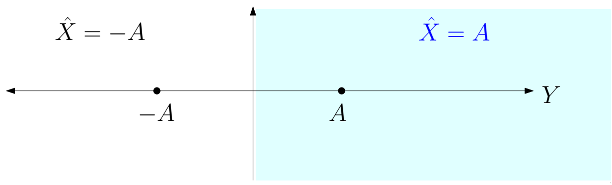


4) 16-QAM: What are the decision regions?



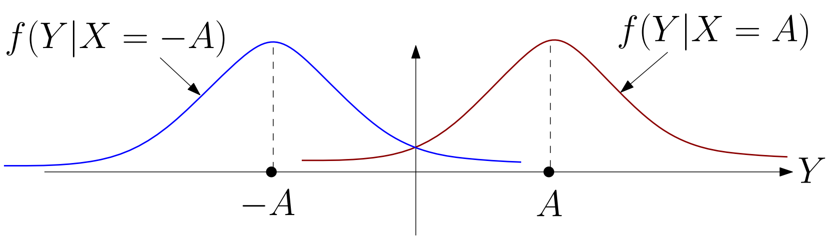
Error Probability for BPSK

Recall the decision regions for BPSK:



Probability of detection error is

$$P_e = P(\hat{X} \neq X) = \frac{1}{2}P(\hat{X} = A \mid X = -A) + \frac{1}{2}P(\hat{X} = -A \mid X = A)$$



$$\begin{aligned} P(\hat{X} = A \mid X = -A) &= P(Y > 0 \mid X = -A) \\ &= P(-A + N > 0 \mid X = -A) \\ &= P(N > A \mid X = -A) = P(N > A) \end{aligned}$$

Error Probability for BPSK ctd.

$$P(N > A) = P\left(\frac{N}{\sqrt{N_0/2}} > \frac{A}{\sqrt{N_0/2}}\right) = \mathcal{Q}\left(\frac{A}{\sqrt{N_0/2}}\right)$$

where

$$\mathcal{Q}(y) = \int_y^\infty \tfrac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

(Note that $\frac{N}{\sqrt{N_0/2}}$ is a $\mathcal{N}(0,1)$ random variable.)

Similarly,

$$\begin{aligned} P(\hat{X} = -A \mid X = A) &= P(A + N < 0 \mid X = A) \\ &= P(N < -A) \\ &= P\left(\frac{N}{\sqrt{N_0/2}} < \frac{-A}{\sqrt{N_0/2}}\right) \\ &= \mathcal{Q}\Big(\frac{A}{\sqrt{N_0/2}}\Big) \quad \text{(symmetry of the Gaussian pdf)} \end{aligned}$$

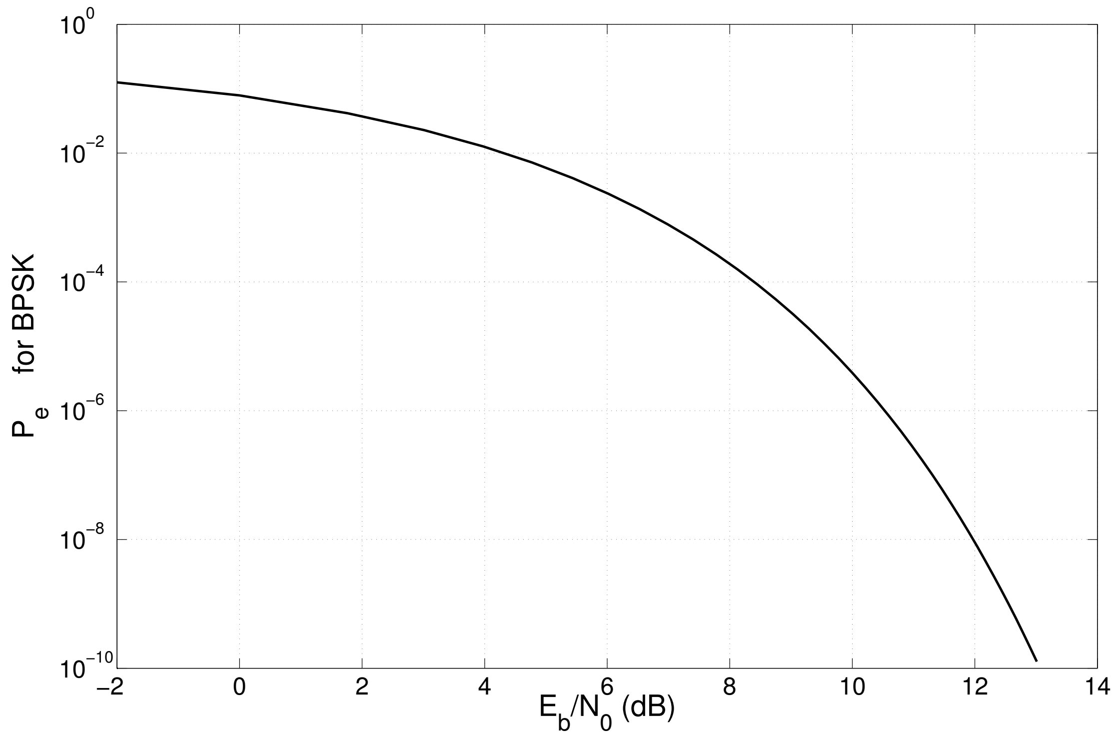
Hence the probability of detection error for BPSK is

$$\begin{aligned} P_e &= \tfrac{1}{2} P(\hat{X} = A \mid X = -A) + \tfrac{1}{2} P(\hat{X} = -A \mid X = A) \\ &= \mathcal{Q}\Big(\frac{A}{\sqrt{N_0/2}}\Big) \\ &\overset{(a)}{=} \mathcal{Q}\Big(\sqrt{\frac{2E_b}{N_0}}\Big) \end{aligned}$$

To obtain (a), observe that:

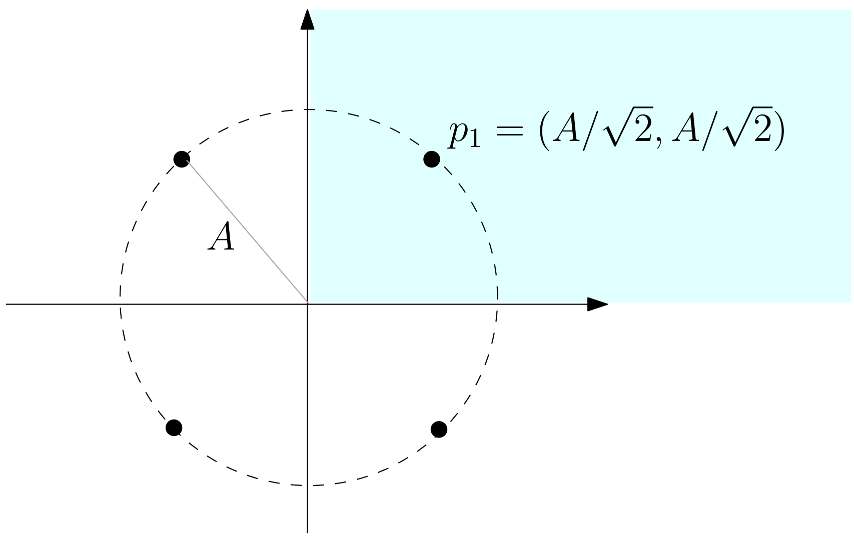
- Energy/symbol $E_s = E_b \log M$, where E_b = energy/bit, M = size of constellation
- For BPSK, $E_s = E_b = A^2$

BPSK Error Probability vs E_b/N_0



To get P_e of 10^{-3}, need *snr* $E_b/N_0 \approx 7$ dB

QPSK Error Probability



You will show in Examples Paper 3 that:

$$
\begin{aligned}
P(\hat{X} \neq p_1 \mid X = p_1) &= P(\{A/\sqrt{2} + N^r < 0\} \cup \{A/\sqrt{2} + N^i < 0\}) \\
&\leq Q(\sqrt{A^2/N_0}) + Q(\sqrt{A^2/N_0}) \\
&= 2Q(\sqrt{E_s/N_0})
\end{aligned}
$$

Demodulation and Detection for M-ary FSK

To transmit message $i \in \{1, \ldots, M\}$ in any symbol period $[kT, (k+1)T)$, the FSK waveform is

$$X(t) = \sqrt{E_s} f_i(t),$$

where $f_i(t) = \sqrt{\frac{2}{T}} \cos\left(2\pi\left(f_c + (2i - (M+1))\frac{\Delta_f}{2}\right) t\right)$.

Recall: $\Delta_f = \frac{1}{2T}$, and the $\{f_i(t)\}_{i=1,\ldots,M}$ form an orthonormal set.

At the receiver, we have

$$Y(t) = X(t) + N(t)$$

Demodulator produces a vector $[Y_1, \ldots, Y_M]$ as

$$\begin{aligned} Y_1 &= < Y(t), f_1(t) > = \int_{t \in \text{symbol period}} Y(t) f_1(t) dt \\ Y_2 &= < Y(t), f_2(t) > \\ &\vdots \\ Y_M &= < Y(t), f_M(t) > \end{aligned}$$

We have:

$$\begin{aligned} Y_1 &= < X(t), f_1(t) > + < N(t), f_1(t) > = \quad X_1 + N_1 \\ Y_2 &= < X(t), f_2(t) > + < N(t), f_2(t) > = \quad X_2 + N_2 \\ &\vdots \\ Y_M &= < X(t), f_M(t) > + < N(t), f_M(t) > = X_M + N_M \end{aligned}$$

- As before, the noise variables $N_j \sim \mathcal{N}(0, \frac{N_0}{2})$ are i.i.d.
- If message $i \in \{1, \ldots, M\}$ is transmitted:

$$X_i = \sqrt{E_s}, \qquad X_j = 0, \; j \neq i$$

Hence $[Y_1, \ldots, Y_i, \ldots, Y_M] = [N_1, \ldots, \sqrt{E_s} + N_i, \ldots, N_M]$

What is the optimal detection rule?

M-ary FSK Detection

Assuming all the messages are equally likely, the optimal rule is to pick the message $\hat{m}$ such that:

$$\hat{m} = \arg\max_{1 \le i \le M} \; P(\mathbf{Y} = \mathbf{y} \mid \mathbf{X} = \mathbf{x}(i))$$

1. $\mathbf{x}(i)$ is the input vector corresponding to message i, i.e., it is the length-M vector with $\sqrt{E_s}$ in position i and 0 elsewhere
2. Conditioned on the input being $\mathbf{x}(i)$, the received vector $\mathbf{y} = [n_1, \ldots, \sqrt{E_s} + n_i, \ldots, n_M]$

$$P(\mathbf{y} \mid \mathbf{x}(i)) = P(y_1 \mid x_1 = 0) \ldots P(y_i \mid x_i = \sqrt{E_s}) \ldots P(y_M \mid x_M = 0)$$

$$= \frac{1}{(\sqrt{\pi N_0})^M} \; e^{-y_1^2/N_0} \ldots e^{-(y_i - \sqrt{E_s})^2/N_0} \ldots e^{-y_M^2/N_0}$$

$$= \frac{1}{(\sqrt{\pi N_0})^M} \; e^{-(y_1^2 + \ldots + y_M^2)/N_0} \; e^{-E_s/N_0} \; e^{2E_s y_i/N_0}$$

The optimal detection rule is thus $\hat{m} = \arg\max_{1 \le i \le M} \; y_i$

Probability of Detection Error

Due to symmetry, the probability of error is the same regardless of which message $i \in \{1, \ldots, M\}$ was transmitted. So, without loss of generality, assume message 1 was transmitted.

An error occurs if y_1 is *not* the maximum among $[y_1, \ldots, y_K] \Rightarrow$

$$P_e = P\left(\{y_1 \le y_2\} \cup \{y_1 \le y_3\} \cup \ldots \cup \{y_1 \le y_M\}\right)$$

$$= P\left(\{\sqrt{E_s} + n_1 \le n_2\} \cup \{\sqrt{E_s} + n_1 \le n_3\} \ldots \cup \{\sqrt{E_s} + n_1 \le n_M\}\right)$$

$$\le P\left(\{\sqrt{E_s} + n_1 \le n_2\}\right) + \ldots + P\left(\{\sqrt{E_s} + n_1 \le n_M\}\right)$$

$$= (M-1) P\left(n_2 - n_1 \ge \sqrt{E_s}\right)$$

The last line holds because $(n_2 - n_1)$, $(n_3 - n_1)$, $\ldots$,$(n_M - n_1)$ are each $\mathcal{N}(0, N_0)$ rvs. In Examples Paper 3, you will simplify the above expression and show that

$$P_e \le e^{-(\log_2 M)(\frac{E_b}{N_0} - 2\ln 2)}, \qquad \text{where } E_s = E_b \log_2 M$$

Thus if $\frac{E_b}{N_0} > 2\ln 2$, the probability of detection error $\downarrow$ as $M \uparrow$

Rate and Bandwidth

Rate: Both M-ary QAM and M-ary FSK have rate $\frac{\log_2 \mathbf{M}}{\mathbf{T}}$ bits/s

Bandwidth:

- **QAM**, the bandwidth is determined by the baseband pulse $p(t)$. If $p(t)$ is band-limited to $[-W, W]$, then $x(t)$ is bandlimited to $[f_c - W, f_c + W]$, and so has bandwidth **2W**. If $p(t)$ is a sinc pulse, $W = \frac{1}{2T}$. For a raised cosine pulse, W is slightly bigger than $\frac{1}{2T}$
- M-**FSK**: For message $i \in \{1, \ldots, M\}$, the signal $x(t)$ is a cosine with frequency $f_c + (i - \frac{(M+1)}{2})\Delta_f$. Hence the total bandwidth required for M-ary FSK is $(M-1)\Delta_f = \frac{(\mathbf{M}-\mathbf{1})}{\mathbf{2T}}$.

The *bandwidth efficiency* η is defined as rate/bandwidth:

$$\eta_{QAM} \approx 2\log_2 M \text{ bits/s/Hz}, \qquad \eta_{MFSK} = \frac{2\log_2 M}{M-1} \text{ bits/s/Hz}$$

As M increases, $\eta_{QAM} \uparrow$ but $\eta_{MFSK} \downarrow$. But how does P_e change with M? (Examples Paper 3)

Summary

Modulation:

$$\text{Bits} \longrightarrow K\text{-dim. vector} \xrightarrow{\text{ortho-basis}} \text{Transmitted waveform } X(t)$$

Demodulation: Project $Y(t)$ onto each basis function to produce $\mathbf{Y} = [Y_1, \ldots, Y_K]$

Detection: Apply MAP rule to determine the transmitted vector (symbol) from $\mathbf{Y} = [Y_1, \ldots, Y_K]$

— When all messages are equally likely, MAP rule becomes a "min-distance" decoding.

Error Analysis: Bounds on probability of detection error for various modulation schemes can be obtained using the Gaussian $\mathcal{Q}$ function and union bounds

You can now do Questions 1–6 in Examples Paper 3.