4F5: Advanced Communications and Coding

Handout 9: Detection in Fading Channels, Diversity

Ramji Venkataraman

Signal Processing and Communications Lab
Department of Engineering
ramji.v@eng.cam.ac.uk

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BPSK on the AWGN channel

$$Y = X + N$$

- BPSK: $X \in \{A, -A\}$
- AWGN $N \sim \mathcal{N}(0, \frac{N_0}{2})$

Probability of detection error $P_e = \mathcal{Q}\Big(\dfrac{A}{\sqrt{N_0/2}}\Big) = \mathcal{Q}(\sqrt{2\,\text{snr}})$

$$\text{snr} = \frac{E_b}{N_0} = \frac{A^2}{N_0}$$

$\mathcal{Q}(x) \leq \frac{1}{2}e^{-x^2/2}$ for $x > 0 \Rightarrow$ P_e *decays exponentially* with snr

BPSK on a Rayleigh Flat Fading Channel

$$Y = hX + N$$
$$h \sim \mathcal{CN}(0,1), \quad N \sim \mathcal{CN}(0, N_0)$$

- *Coherent* detection: The complex fading coefficient h can be estimated at the receiver by transmitting a known sequence (called a pilot or training sequence)
- The receiver multiplies Y by $\frac{h^*}{|h|}$ to obtain $\bar{Y}$:

$$\bar{Y} = |h|\, X + \bar{N} \tag{1}$$

  where $\bar{N} = \frac{h^*}{|h|} N$ is a $\mathcal{CN}(0, N_0)$ random variable (why?)
- When h is known, BPSK with $X \in \pm A$ on the channel (1) is equivalent to a no-fading AWGN channel with BPSK $\pm|h|A$
- As signal is real, only real part of $\bar{Y}$ matters for detection.
  $\Rightarrow$ *Scalar* BPSK detection with noise $\text{Re}[\bar{N}] \sim \mathcal{N}(0, \frac{N_0}{2})$

Probability of Error for BPSK

Thus the probability of error *conditioned* on h is

$$P_{e|h} = \mathcal{Q}\Big(\frac{|h|A}{\sqrt{N_0/2}}\Big) = \mathcal{Q}\Big(\sqrt{2|h|^2 \text{snr}}\Big)$$

The *average* probability of error with coherent detection is

$$P_e = \mathbb{E}\left[\mathcal{Q}\Big(\sqrt{2|h|^2 \text{snr}}\Big)\right]$$

where the expectation is over $|h|^2$.

$h \sim \mathcal{CN}(0,1) \;\Rightarrow\; |h|^2$ has an *exponential* density (Handout 8):

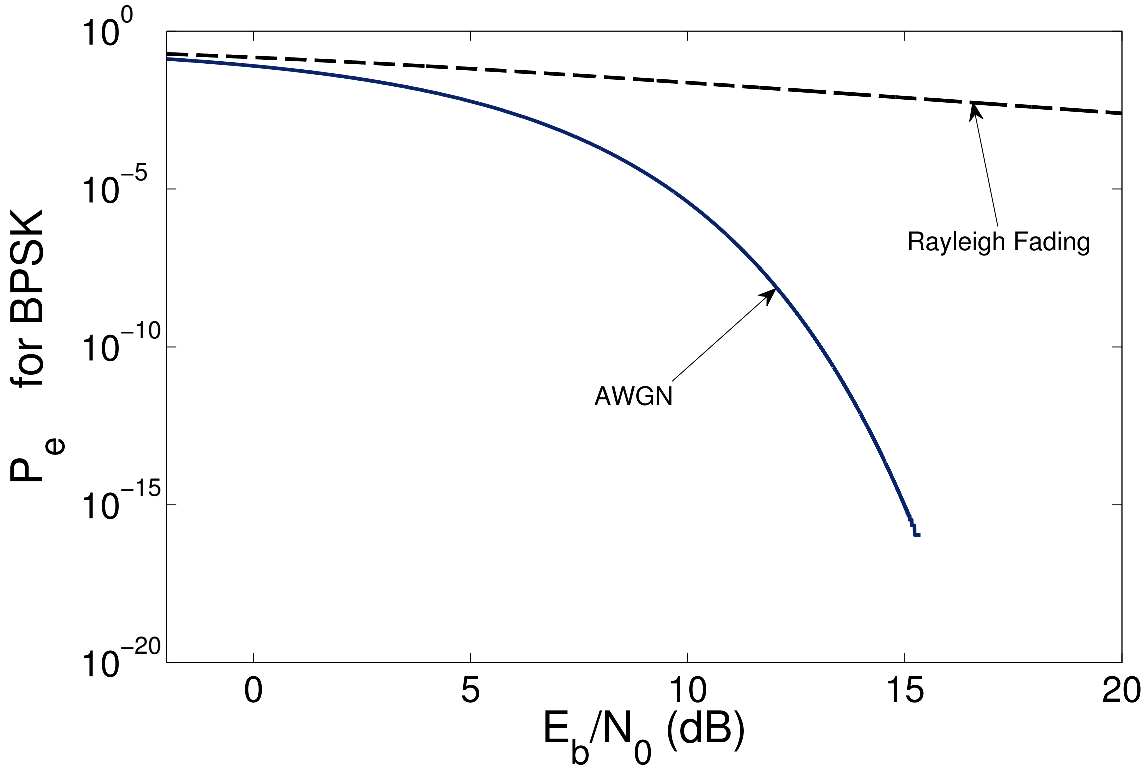
$$f_{|h|^2}(x) = \exp(-x), \quad x \geq 0$$

Therefore

$$P_e = \int_0^\infty \exp(-x)\mathcal{Q}\Big(\sqrt{2x\,\text{snr}}\Big)\, dx \overset{(a)}{=} \frac{1}{2}\Big(1 - \sqrt{\frac{\text{snr}}{1+\text{snr}}}\Big)$$

You will show (a) in Examples Paper 3, Q.7

Error Performance on AWGN vs Flat Fading



For $P_e = 10^{-3}$, there is a 17dB difference in required snr between AWGN and Rayleigh fading !

For large snr, we can approximate

$$\sqrt{\frac{\mathsf{snr}}{1+\mathsf{snr}}} = \left(1 + \frac{1}{\mathsf{snr}}\right)^{-1/2} \approx 1 - \frac{1}{2\,\mathsf{snr}} + O\left(\frac{1}{\mathsf{snr}^2}\right)$$

Thus the average probability of error for BPSK is

$$P_e = \frac{1}{2}\left(1 - \sqrt{\frac{\mathsf{snr}}{1+\mathsf{snr}}}\right) \approx \frac{1}{4\,\mathsf{snr}} \quad \text{for large snr}$$

- P_e decays inversely proportional to snr
- This decay is *much slower* than exponential decay of P_e without fading

Deep Fades

$$P_e \approx \frac{1}{4\,\text{snr}} \text{ for large snr}$$

Why does BPSK perform so poorly over a fading channel *even* with coherent detection (i.e., we know h at the receiver) ?

Can answer this by looking at the high snr regime:

- Error probability conditioned on h is $P_{e|h} = \mathcal{Q}\left(\sqrt{2|h|^2\text{snr}}\right)$
- When $|h|^2 \gg \frac{1}{\text{snr}}$, $P_{e|h}$ is very small (the good case)
- When $|h|^2 < \frac{1}{\text{snr}}$, $P_{e|h}$ is large (bad!)

$$P\left(|h|^2 < \frac{1}{\text{snr}}\right) = \int_0^{1/\text{snr}} e^{-x}dx = \frac{1}{\text{snr}} + O\left(\frac{1}{\text{snr}}\right)^2$$

The dominant source of error is the channel being in *deep fade* :

- Deep fade is defined as the event that $|h|^2 < \frac{1}{\text{snr}}$
- $P(\text{deep fade}) \approx \frac{1}{\text{snr}}$

Diversity

Deep fades are inevitable in wireless channels.

Main idea to combat deep fades and improve P_e: Diversity

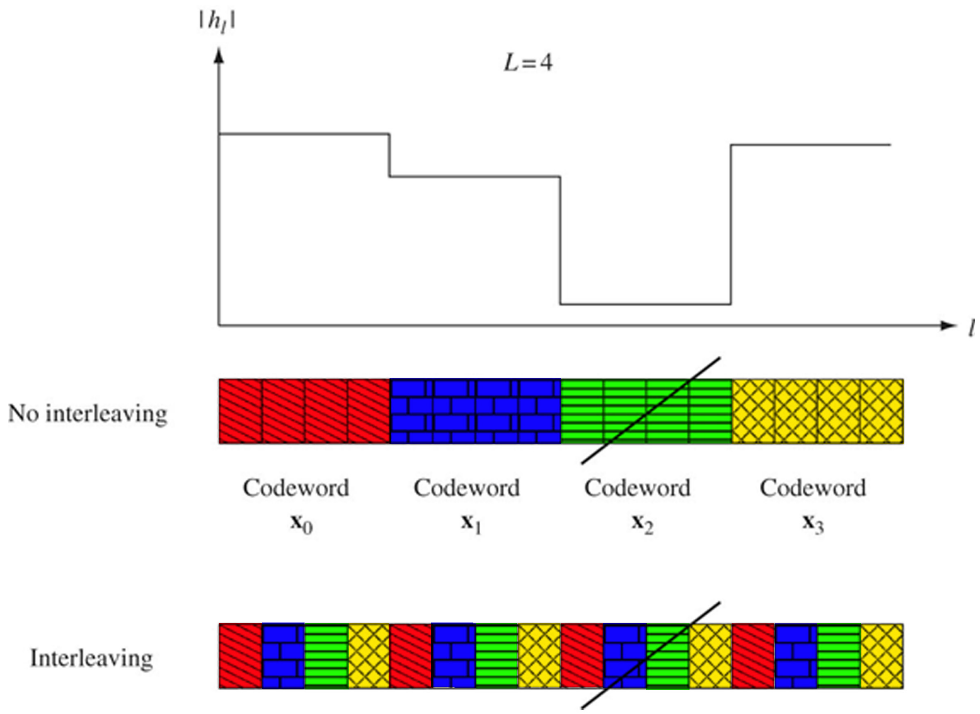
- Transmit information symbols through multiple *independently* faded signal paths
- Reliable communication possible as long as one of the paths is strong

Ways to obtain diversity:

1. **Time** diversity via *coding* and *interleaving*
2. **Frequency** diversity if channel is frequency-selective (multiple channel taps)
3. **Space** diversity with multiple transmit/receive antennas spaced sufficiently apart

We now discuss a simple time diversity scheme

Time Diversity



From "Fundamentals of Wireless Communication", Tse and Viswanath, CUP 2005

Interleaving ensures that each symbol of every codeword experiences an independent fade.

Time Diversity via Repetition

Consider a flat fading channel used L times after *interleaving* over L coherence time periods:

$$Y[m] = h[m]X[m] + N[m], \qquad \text{for } m = 1, \ldots, L$$

Interleaving ensures that that $h[1], \ldots, h[m]$ are i.i.d $\sim \mathcal{CN}(0,1)$

Repetition Coding:

$$X[m] = x \qquad \text{for } m = 1, \ldots, L$$

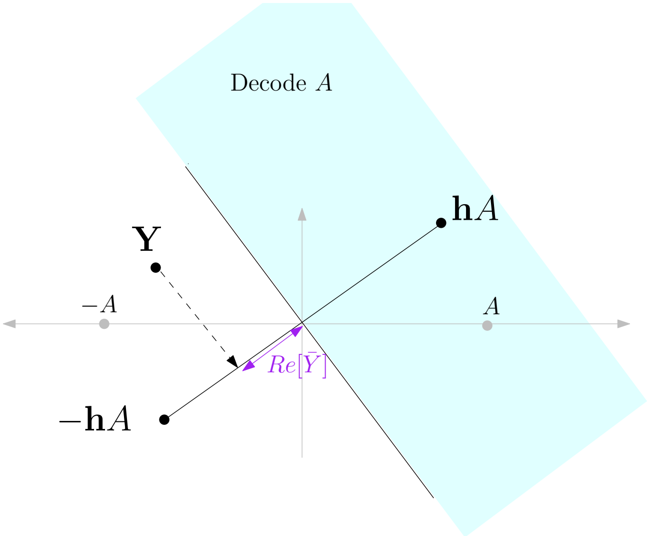
In vector form, the output $\mathbf{Y} = (Y[1], \ldots, Y[L])^T$ is

$$\mathbf{Y} = \mathbf{h}x + \mathbf{N}$$

- $\mathbf{h} = (h[1], \ldots, h[L])^T$ are i.i.d. $\sim \mathcal{CN}(0,1)$
- $\mathbf{N} = (N[1], \ldots, N[L])^T$ are i.i.d $\sim \mathcal{CN}(0, N_0)$

For $x \in \{+A, -A\}$ (BPSK):
What is the optimum detector? Average P_e?

Vector Detection in Gaussian Noise



Project the output $\mathbf{Y}$ in the direction of $\mathbf{h}$:

$$\bar{Y} = \frac{\mathbf{h}^*}{\|\mathbf{h}\|}\mathbf{Y} = \|\mathbf{h}\|x + \bar{N} \qquad \text{where} \quad \|\mathbf{h}\|^2 = \sum_{m=1}^{L} |h[m]|^2$$

- $\bar{N}$ is $\mathcal{CN}(0, N_0)$ (linear combination of L i.i.d $\mathcal{CN}(0, N_0)$ rvs)
- The signal is *real*: either $+\|\mathbf{h}\|A$ or $-\|\mathbf{h}\|A$
  $\Rightarrow$ *Scalar* BPSK detection from $\text{Re}[\bar{Y}]$

Probability of Detection Error

$$P_{e|h} = \mathcal{Q}\Big(\frac{\|\mathbf{h}\|A}{\sqrt{N_0/2}}\Big) = \mathcal{Q}\Big(\sqrt{2\|\mathbf{h}\|^2\text{snr}}\Big)$$

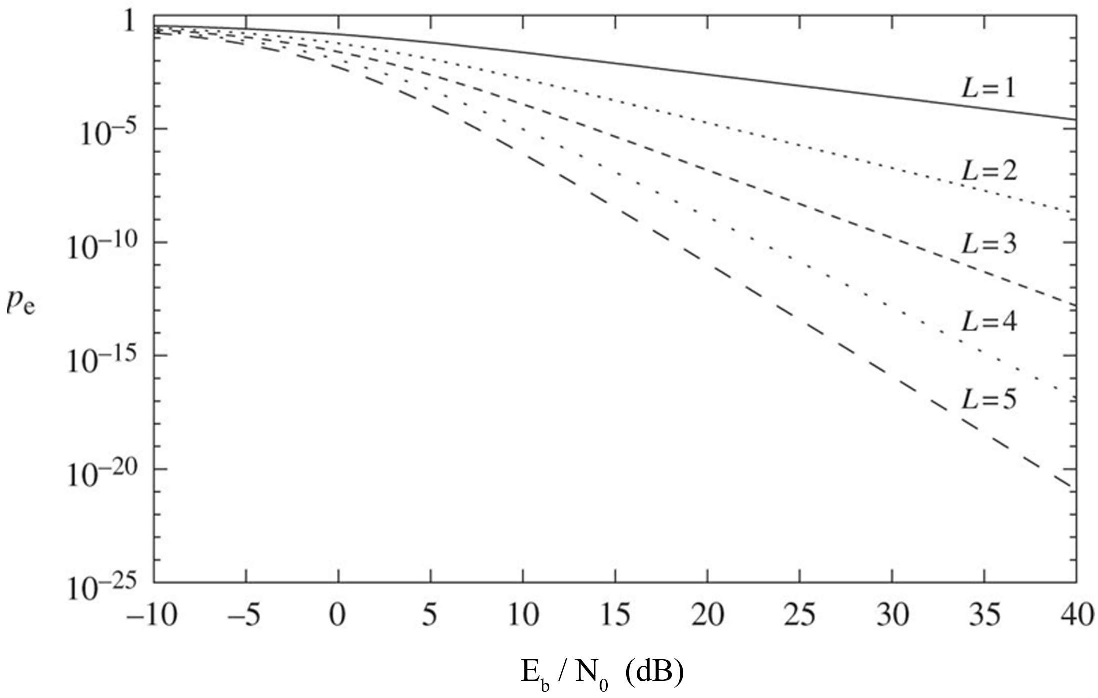
The rv $\|\mathbf{h}\|^2 = \sum_{m=1}^{L} |h[m]|^2$ is:

- The sum of squared-magnitudes of L i.i.d. complex rvs, each $\mathcal{CN}(0, 1)$
- Chi-squared distributed with $2L$ degrees of freedom. The density f of $\|\mathbf{h}\|^2$ is

$$f(x) = \frac{1}{(L-1)!}x^{L-1}e^{-x}, \quad x \geq 0$$

Average $P_e = \mathbb{E}\Big[\ \mathcal{Q}\Big(\sqrt{2\|\mathbf{h}\|^2\text{snr}}\Big)\ \Big]$ where the $\mathbb{E}$ is w.r.t $\|\mathbf{h}\|^2$

Average P_e with Repetition Coding



From "Fundamentals of Wireless Communication", Tse and Viswanath, CUP 2005

Diversity gain with $L = 4$ or 5 is significant!

Pr(deep fade)

The probability of deep fade for large snr is

$$\Pr(\|\mathbf{h}\|^2 < 1/\text{snr}) = \int_0^{1/\text{snr}} \frac{1}{(L-1)!} x^{L-1} e^{-x}\, dx$$

$$\overset{(a)}{\approx} \int_0^{1/\text{snr}} \frac{1}{(L-1)!} x^{L-1}\, dx = \frac{1}{L!} \frac{1}{\text{snr}^L}$$

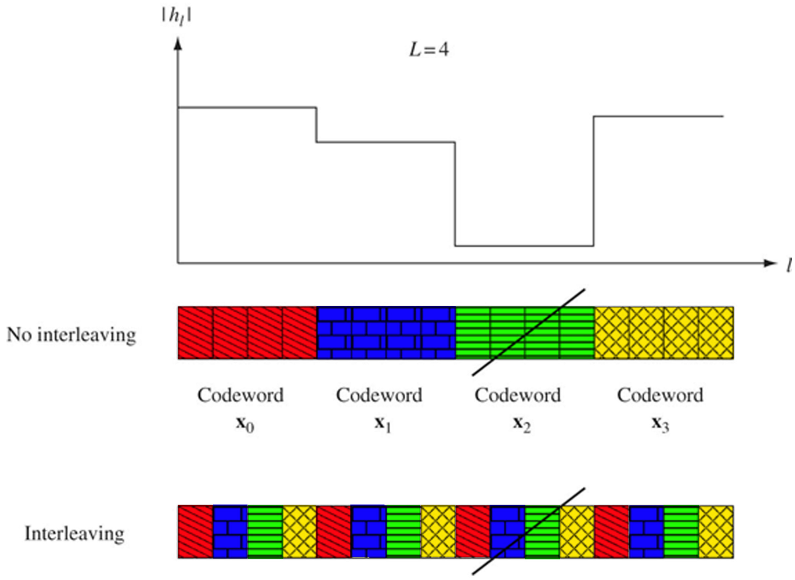
(a) holds because for small x, $e^{-x} \approx 1$

- Pr(deep fade) falls as *Lth power of snr*
- Can show that the average probability of error is

$$P_e = \int_0^{\infty} \mathcal{Q}(\sqrt{2\, x\, \text{snr}}) f(x) dx \approx \binom{2L-1}{L} \frac{1}{(4\, \text{snr})^L}$$

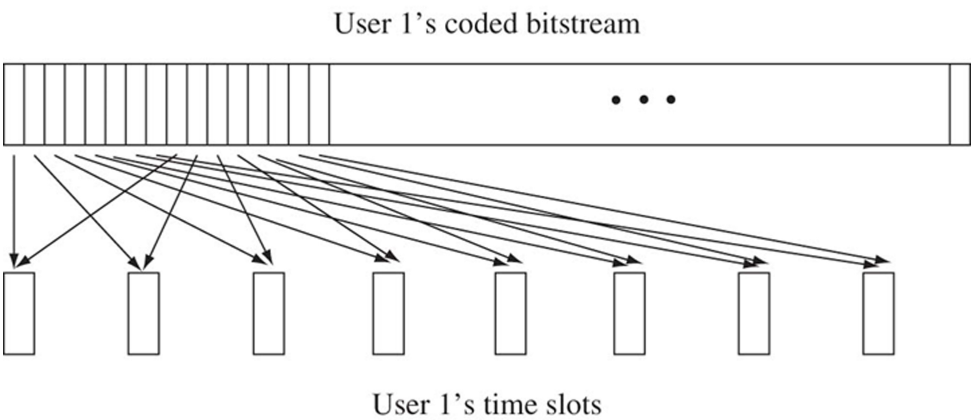
Beyond Repetition Coding

- Repeating each symbol L times reduces rate by factor of $1/L$
- Can use a more sophisticated block code than repetition
- Can get diversity gain with much better rates



From "Fundamentals of Wireless Communication", Tse and Viswanath, CUP 2005

Example: GSM



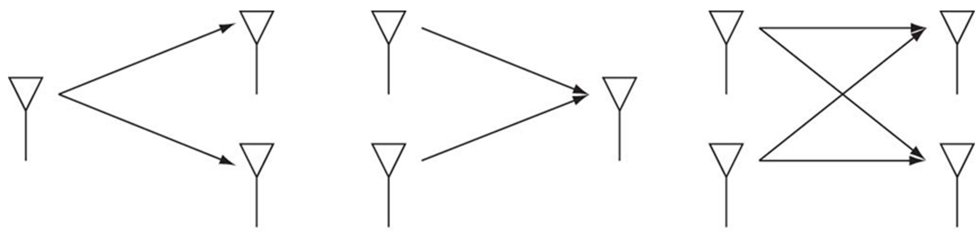
From "Fundamentals of Wireless Communication", Tse and Viswanath, CUP 2005

- Amount of time diversity limited by two things:
  - How fast channel varies (coherence time T_c)
  - Delay constraint of the application
- In GSM (voice), delay constraint is 40ms.

Other kinds of Diversity

Key Q: How to get multiple signal paths that fade independently?

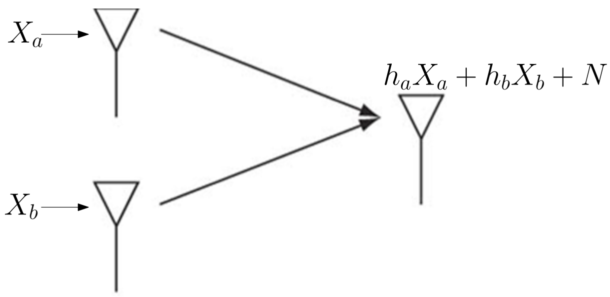
Antenna Diversity: Multiple antennas spaced sufficiently apart give independent signal paths



From "Fundamentals of Wireless Communication", Tse and Viswanath, CUP 2005

- Cheap to have multiple antennas at the base station, more challenging in a handset
- We'll now look at some signalling schemes with two Tx antennas and 1 Rx antenna

Transmit Antenna Diversity



With two Tx and one Rx antenna, the channel is

$$Y = h_a X_a + h_b X_b + N.$$

- h_a, h_b are the channel fading coefficients from antennas 1 and 2, respectively. h_a, h_b are iid $\sim \mathcal{CN}(0,1)$.
- X_a, X_b are the symbols transmitted from antennas 1 and 2, respectively.
- The additive noise $N \sim \mathcal{CN}(0, N_0)$.

Repetition coding with antennas

- In the first time period, antenna 1 transmits symbol x, and antenna 2 remains silent.
- In the next time period, antenna 2 transmits the same symbol x, and antenna 1 remains silent.

So we have

$$\mathbf{Y} = \mathbf{h}x + \mathbf{N}$$

$\mathbf{Y} = [Y[1], Y[2]]^T, \quad \mathbf{h} = [h_a, h_b]^T \sim$ iid $\mathcal{CN}(0,1)$
$\mathbf{N} = [N[1], N[2]]^T \sim$ iid $\mathcal{CN}(0, N_0)$

Multiply $\mathbf{Y}$ by $\frac{\mathbf{h}^*}{|h|}$ to get $\bar{Y} = \|\mathbf{h}\|x + \bar{N}$ . . . this is exactly the same as the time-diversity scheme with repetition coding (see p.11)

We get diversity gain of $L = 2 \;\Rightarrow\; P_e \sim \frac{1}{snr^2}$

Rate: 1 constellation symbol every *two* time periods. It seems wasteful for one antenna to stay silent while the other is active.

Can we do better? Can we transmit *two* constellation symbols in two periods, and still get $P_e \sim \frac{1}{snr^2}$?

The Alamouti Scheme

Transmit *two* symbols u and v in two periods as follows.
Symbol time 1: Antenna 1 transmits u, antenna 2 transmits v
Symbol time 2: Antenna 1 transmits $-v^*$, antenna 2 transmits u^*

$$\begin{aligned} Y[1] &= \quad h_a u + h_b v + N[1] \\ Y[2] &= -h_a v^* + h_b u^* + N[2] \end{aligned}$$

At the Rx, conjugate the second output and rewrite as:

$$\begin{aligned} Y[1] &= \; h_a u + h_b v + N[1] \\ Y[2]^* &= \; h_b^* u - h_a^* v + N[2]^* \end{aligned} \qquad (2)$$

Key observation: The channel coefficient vector $[h_a[1], h_b^*[2]]$ multiplying u is *orthogonal* to the vector $[h_b^*, -h_a^*]$ multiplying v:

$$[\, h_a, h_b^*]^* \begin{bmatrix} h_b^* \\ -h_a^* \end{bmatrix} = 0$$

This gives us a nice way to detect u and v separately from (2).

Let $\|h\| = \sqrt{|h_a|^2 + |h_b|^2}$. At the Rx:
1) Take inner product of the vector $[Y[1], Y[2]^*]$ with the vector $\frac{1}{\|h\|}[h_a^*, h_b]$ to get:

$$\bar{Y} = \|h\| u + \bar{N} \qquad (3)$$

2) Take inner product of the vector $[Y[1], Y[2]^*]$ with the vector $\frac{1}{\|h\|}[h_b^*, -h_a]$ to get:

$$\bar{Y}' = \|h\| v + \bar{N}' \qquad (4)$$

Can now detect u from $\bar{Y}$ and v from $\bar{Y}'$ *separately*, without interference from the other symbol.
The noise rvs $\bar{N}$ and $\bar{N}'$ are each $\mathcal{CN}(0, N_0)$

- Note that detecting u from (3), and v from (4) gives the same diversity gain as repetition coding with $L = 2$.
- But we have transmitted two symbols in two time periods — double the rate!
- The Alamouti scheme is a simple example of a "space-time" code, where the information is coded across both space (antennas) and time (symbol periods).

Frequency Diversity

If we have a large bandwidth W, you get multiple channel taps

$$Y[m] = \sum_{\ell} h_\ell X[m-\ell] + N[m], \quad m = 1, 2, \ldots$$

The h_ℓ's are independent; but we have to deal with ISI

Main challenge: How to mitigate ISI while exploiting the diversity in the frequency-selective channel?

- Direct-sequence spread spectrum (e.g. IS-95 CDMA)
- Orthogonal frequency-division multiplexing (OFDM):
  - Channel acts like a filter with impulse response $\{h_\ell\}$
  - In freq. domain, signal DFT $\tilde{x}[m]$ gets **multiplied** with $\tilde{h}_m$:

    $$\tilde{y}[k] = \tilde{h}_k \tilde{x}[k] + \tilde{n}[k]$$

    No ISI in freq. domain! Code information as DFT symbols, transmit *Inverse DFT* over the channel
  - OFDM with QPSK/QAM constellation used in 3G/LTE as well as Wi-Fi 802.11n

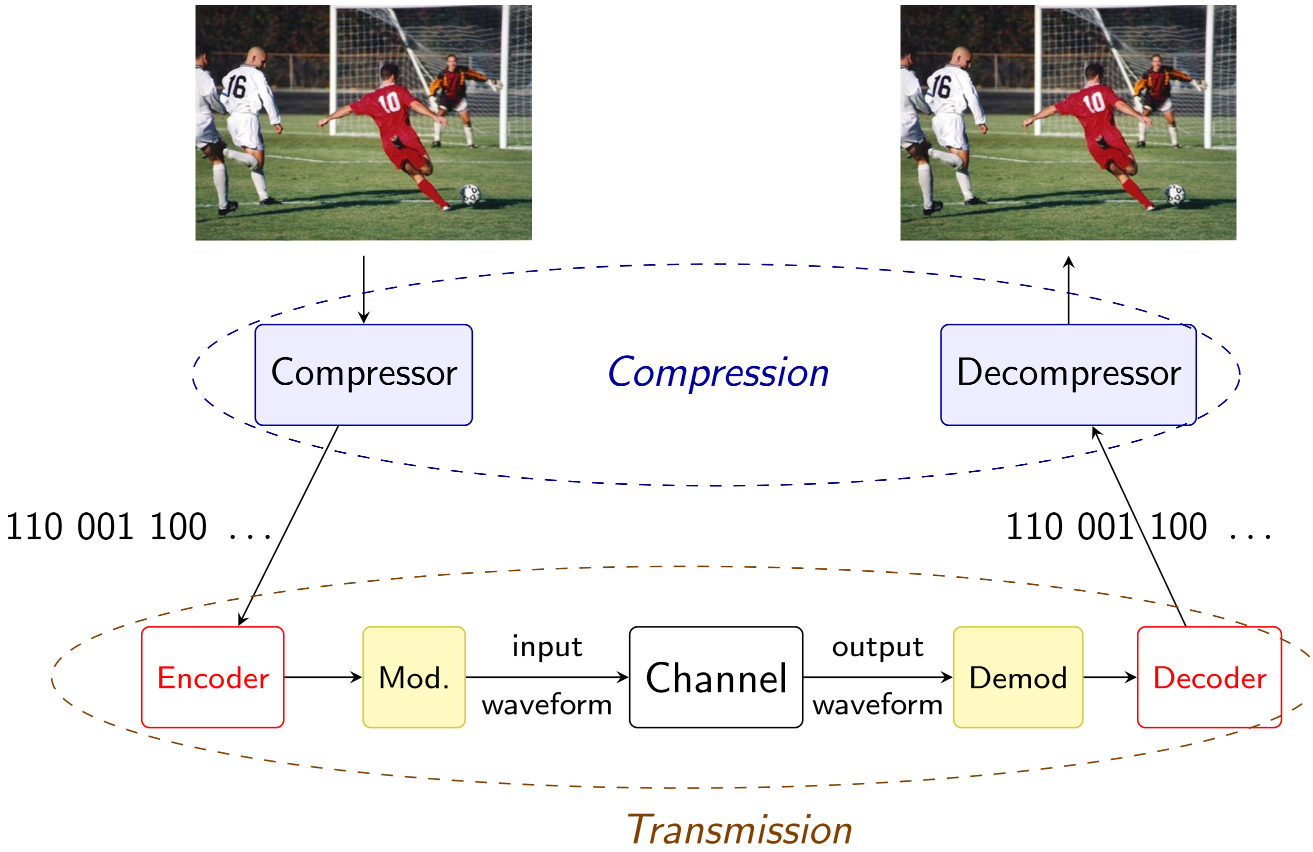
Wireless Channels: A Summary

- Fading makes wireless channels unreliable
- Diversity increases reliability and makes the channel more consistent
- Good codes in conjunction with diversity techniques (time/space/frequency) yield a coding gain in addition to diversity gain

For more on diversity techniques and wireless system design, see

Fundamentals of Wireless Communication, Tse & Viswanath, CUP 2005.
Available at: `http://www.eecs.berkeley.edu/~dtse/book.html`

Review: End-to-End Communication System



Course summary

1. **Information Representation, Compression**:

   - Sources and channels are modelled using probability

   - Entropy, Conditional Entropy, Typical Sequences; how is all this relevant to compression?

   - Mutual Information; Capacity, the fundamental limit of data transmission

2. **Channel Coding**: Next we studied how to design practical encoders and decoders:

   - How to achieve channel capacity with *low-complexity* codes?

   - Reed-Solomon, LDPC codes (almost every communication or storage device has one of these)

3. **Modulation Techniques and Wireless Channels**:

   We looked at the inner "physical" part of the Tx/Rx system: Modelling real-world channels; how to effectively transmit bits/symbols using waveforms

Exam Information

- 1.5 hours, 4 questions out of which you do 3.
- One question from each of the three parts of the course; the fourth may be a combination or from one of the parts.
- There will be a data sheet attached to the exam. This sheet will be available on Moodle in early January.
- A list of relevant past Tripos questions will also be available on Moodle.
- We will have an examples class early in Lent Term
  — on **15 January at 4pm**