4F5: Advanced Communications and Coding
Handout 10: Detection in Fading Channels, Diversity

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BPSK on the AWGN channel

\[ Y = X + N \]

- BPSK: \( X \in \{A, -A\} \)
- AWGN: \( N \sim \mathcal{N}(0, \frac{N_0}{2}) \)

Probability of detection error \( P_e = Q\left(\frac{A}{\sqrt{N_0/2}}\right) = Q(\sqrt{2}\text{snr}) \)

\[ \text{snr} = \frac{E_b}{N_0} = \frac{A^2}{N_0} \]

\( Q(x) \leq \frac{1}{2} e^{-x^2/2} \) for \( x > 0 \) \( \Rightarrow \) \( P_e \) decays exponentially with \( \text{snr} \)
BPSK on a Rayleigh Flat Fading Channel

\[ Y = hX + N \]

\[ h \sim \mathcal{CN}(0, 1), \quad N \sim \mathcal{CN}(0, N_0) \]

- **Coherent** detection: The complex fading coefficient \( h \) can be estimated at the receiver by transmitting a known sequence (called a pilot or training sequence)
- The receiver multiplies \( Y \) by \( \frac{h^*}{|h|} \) to obtain \( \bar{Y} \):
  \[ \bar{Y} = |h|X + \bar{N} \tag{1} \]
  where \( \bar{N} = \frac{h^*}{|h|}N \) is a \( \mathcal{CN}(0, N_0) \) random variable (why?)
- When \( h \) is known, BPSK with \( X \in \pm A \) on the channel (1) is equivalent to a no-fading AWGN channel with BPSK \( \pm |h|A \)
- As signal is real, only real part of \( \bar{Y} \) matters for detection.
  \[ \Rightarrow Scalar \text{ BPSK detection with noise } \text{Re}[\bar{N}] \sim \mathcal{N}(0, \frac{N_0}{2}) \]
Probability of Error for BPSK

Thus the probability of error *conditioned* on $h$ is

$$P_{e|h} = Q\left(\frac{|h|A}{\sqrt{N_0/2}}\right) = Q\left(\sqrt{2|h|^2\text{snr}}\right)$$

The *average* probability of error with coherent detection is

$$P_e = \mathbb{E}\left[Q\left(\sqrt{2|h|^2\text{snr}}\right)\right]$$

where the expectation is over $|h|^2$.

$h \sim \mathcal{CN}(0, 1) \Rightarrow |h|^2$ has an *exponential* density (Handout 9):

$$f_{|h|^2}(x) = \exp(-x), \quad x \geq 0$$

Therefore

$$P_e = \int_0^\infty \exp(-x)Q\left(\sqrt{2x \text{snr}}\right) \, dx \overset{(a)}{=} \frac{1}{2} \left(1 - \sqrt{\frac{\text{snr}}{1 + \text{snr}}}\right)$$

You will show *(a)* in Examples Paper 2, Q.8
Error Performance on AWGN vs Flat Fading

For $P_e = 10^{-3}$, there is a 17dB difference in required snr between AWGN and Rayleigh fading!
For large snr, we can approximate

\[
\sqrt{\frac{\text{snr}}{1 + \text{snr}}} = \left(1 + \frac{1}{\text{snr}}\right)^{-1/2} \approx 1 - \frac{1}{2\text{snr}} + O\left(\frac{1}{\text{snr}^2}\right)
\]

Thus the average probability of error for BPSK is

\[
P_e = \frac{1}{2} \left(1 - \sqrt{\frac{\text{snr}}{1 + \text{snr}}}\right) \approx \frac{1}{4\text{snr}} \quad \text{for large snr}
\]

- \(P_e\) decays inversely proportional to snr
- This decay is *much slower* than exponential decay of \(P_e\) without fading
Deep Fades

\[ P_e \approx \frac{1}{4\text{snr}} \text{ for large snr} \]

Why does BPSK perform so poorly over a fading channel even with coherent detection (i.e., we know \( h \) at the receiver)?

Can answer this by looking at the high snr regime:

- Error probability conditioned on \( h \) is \( P_e|h = Q(\sqrt{2|h|^2\text{snr}}) \)
- When \( |h|^2 \gg \frac{1}{\text{snr}} \), \( P_e|h \) is very small (the good case)
- When \( |h|^2 < \frac{1}{\text{snr}} \), \( P_e|h \) is large (bad!)

\[
P \left( |h|^2 < \frac{1}{\text{snr}} \right) = \int_0^{1/\text{snr}} e^{-x} dx = \frac{1}{\text{snr}} + O\left(\frac{1}{\text{snr}}\right)^2
\]

The dominant source of error is the channel being in deep fade:

- Deep fade is defined as the event that \( |h|^2 < \frac{1}{\text{snr}} \)
- \( P(\text{deep fade}) \approx \frac{1}{\text{snr}} \)
Deep fades are inevitable in wireless channels.

Main idea to combat deep fades and improve $P_e$: **Diversity**

- Transmit information symbols through multiple *independently* faded signal paths
- Reliable communication possible as long as one of the paths is strong

Ways to obtain diversity:

1. **Time** diversity via *coding* and *interleaving*
2. **Frequency** diversity if channel is frequency-selective (multiple channel taps)
3. **Space** diversity with multiple transmit/receive antennas spaced sufficiently

We will now discuss a simple time diversity scheme
Interleaving ensures that each symbol of every codeword experiences an independent fade.

From “Fundamentals of Wireless Communication”, Tse and Viswanath, CUP 2005
Time Diversity via Repetition

Consider a flat fading channel used $L$ times after *interleaving* over $L$ coherence time periods:

$$Y[m] = h[m]X[m] + N[m], \quad \text{for } m = 1, \ldots, L$$

Interleaving ensures that that $h[1], \ldots, h[m]$ are i.i.d $\sim \mathcal{CN}(0, 1)$

**Repetition Coding:**

$$X[m] = x \quad \text{for } m = 1, \ldots, L$$

In vector form, the output $Y = (Y[1], \ldots, Y[L])^T$ is

$$Y = hx + N$$

- $h = (h[1], \ldots, h[L])^T$ are i.i.d. $\sim \mathcal{CN}(0, 1)$
- $N = (N[1], \ldots, N[L])^T$ are i.i.d $\sim \mathcal{CN}(0, N_0)$

For $x \in \{+A, -A\}$ (BPSK):

What is the optimum detector? Average $P_e$?
Vector Detection in Gaussian Noise

Project the output $\mathbf{Y}$ in the direction of $\mathbf{h}$:

$$\bar{\mathbf{Y}} = \frac{\mathbf{h}^*}{\|\mathbf{h}\|} \mathbf{Y} = \|\mathbf{h}\| \mathbf{x} + \bar{\mathbf{N}}$$

where $\|\mathbf{h}\| = \sum_{m=1}^{L} |h[m]|^2$

- $\bar{\mathbf{N}}$ is $\mathcal{N}(0, \frac{N_0}{2})$ (linear combination of $L$ i.i.d $\mathcal{CN}(0, N_0)$ rvs)

- The signal is real: either $+\|\mathbf{h}\|A$ or $-\|\mathbf{h}\|A$

$\Rightarrow$ Scalar BPSK detection from $\text{Re}[\bar{\mathbf{Y}}]$
Probability of Detection Error

\[ P_{e|h} = Q\left( \frac{\|h\|A}{\sqrt{N_0/2}} \right) = Q\left( \sqrt{2\|h\|^2 \text{snr}} \right) \]

The rv \( \|h\|^2 = \sum_{m=1}^{L} |h[m]|^2 \) is:

- The sum of magnitudes of \( L \) i.i.d. complex rvs, each \( \mathcal{CN}(0,1) \)
- Chi-squared distributed with \( 2L \) degrees of freedom. The density \( f \) of \( \|h\|^2 \) is

\[ f(x) = \frac{1}{(L-1)!} x^{L-1} e^{-x}, \quad x \geq 0 \]

Average \( P_e = \mathbb{E} \left[ Q\left( \sqrt{2\|h\|^2 \text{snr}} \right) \right] \) where the \( \mathbb{E} \) is w.r.t \( \|h\|^2 \)
Average $P_e$ with Repetition Coding

Diversity gain with $L = 4$ or $5$ is significant!

From “Fundamentals of Wireless Communication”, Tse and Viswanath, CUP 2005
The probability of deep fade for large snr is

\[
\Pr(\|h\|^2 < 1/\text{snr}) = \int_0^{1/\text{snr}} \frac{1}{(L-1)!} x^{L-1} e^{-x} \, dx
\]

\[(a) \approx \int_0^{1/\text{snr}} \frac{1}{(L-1)!} x^{L-1} \, dx = \frac{1}{L!} \frac{1}{\text{snr}^L}\]

(a) holds because for small \(x\), \(e^{-x} \approx 1\)

- \(\Pr(\text{deep fade})\) falls as \(L\text{th power of snr}\)
- Can show that the average probability of error is

\[
P_e = \int_0^{\infty} Q(\sqrt{2x\text{snr}})f(x) \, dx \approx \left(\frac{2L - 1}{L}\right) \frac{1}{(4\text{snr})^L}\]
Beyond Repetition Coding

- Repeating each symbol $L$ times reduces rate by factor of $1/L$
- Can use a more sophisticated block code than repetition
- Can get diversity gain with much better rates

From “Fundamentals of Wireless Communication”, Tse and Viswanath, CUP 2005
Amount of time diversity limited by two things:
- How fast channel varies (coherence time $T_c$)
- Delay constraint of the application

In GSM (voice), delay constraint is 40ms.
Other kinds of Diversity

Key Q: How to get multiple signal paths that fade independently?

- **Antenna Diversity**: Multiple antennas spaced sufficiently apart give independent signal paths

- **Frequency Diversity**: If we have a large bandwidth $W$, you get multiple channel taps

$$Y[m] = \sum_{\ell} h_{\ell} X[m - \ell] + N[m], \quad m = 1, 2, \ldots$$

The $h_{\ell}$’s are independent; but we have to deal with *ISI*
Approaches to Frequency Diversity

\[ Y[m] = \sum_{\ell} h_\ell X[m - \ell] + N[m], \quad m = 1, 2, \ldots \]

Main challenge: How to mitigate ISI while exploiting the diversity in the frequency-selective channel?

- Direct-sequence spread spectrum (e.g. IS-95 CDMA)
- Orthogonal frequency-division multiplexing (OFDM):
  - Channel acts like a filter with impulse response \( \{h_\ell\} \)
  - In freq. domain, signal DFT \( \tilde{x}[m] \) gets \textbf{multiplied} with \( \tilde{h}[m] \):
    \[ \tilde{y}[k] = \tilde{h}_k \tilde{x}[k] + \tilde{n}[k] \]
    No ISI in freq. domain! Code information as DFT symbols, transmit \textit{Inverse DFT} over the channel
- OFDM with QPSK/QAM constellation used in 3G/LTE as well as Wi-Fi 802.11n
Wireless Channels: A Summary

- Fading makes wireless channels unreliable
- Diversity increases reliability and makes the channel more consistent
- Good codes in conjunction with diversity techniques (time/space/frequency) yield a coding gain in addition to diversity gain

For more on diversity techniques and wireless system design, see Fundamentals of Wireless Communication, Tse & Viswanath, CUP 2005. Available at: http://www.eecs.berkeley.edu/~dtse/book.html
Review: End-to-End Communication System

Compression

Decompression

Transmission
Course summary

1. **Information Representation, Compression:**
   - Sources and channels are modelled using probability
   - Entropy, Conditional Entropy, Typical Sequences; how is all this relevant to compression?
   - Mutual Information; Capacity, the fundamental limit of data transmission

2. **Modulation Techniques and Wireless Channels:**
   We studied the inner “physical” part of the Tx/Rx system:
   Modelling real-world channels; how to effectively transmit bits/symbols using waveforms

3. **Channel Coding:** Next you’ll look at how to design encoders and decoders:
   - How to achieve channel capacity with *low-complexity* codes?
   - Reed-Solomon, Convolutional, LDPC codes (almost every communication/storage device has one of these)
Exam Information

- 1.5 hours, 4 questions out of which you do 3.

- One question from each of the three parts of the course; the fourth may be a combination or from one of the parts.

- There will be a data sheet attached to the exam. This sheet will be available on Moodle in early January.

- If there is interest, we will have an examples class early in Lent Term (before lectures begin).

- I will be available in BE3-12 on Tuesdays this term from 1:30-3:00. If you want to meet some other time, send an email.