4F5 Advanced Wireless Communications, 2011

Examples Paper I

February 17, 2012

Question 1

Consider the four waveforms $x_m(\cdot), m = 1, \ldots, 4$ shown in Figure 1.

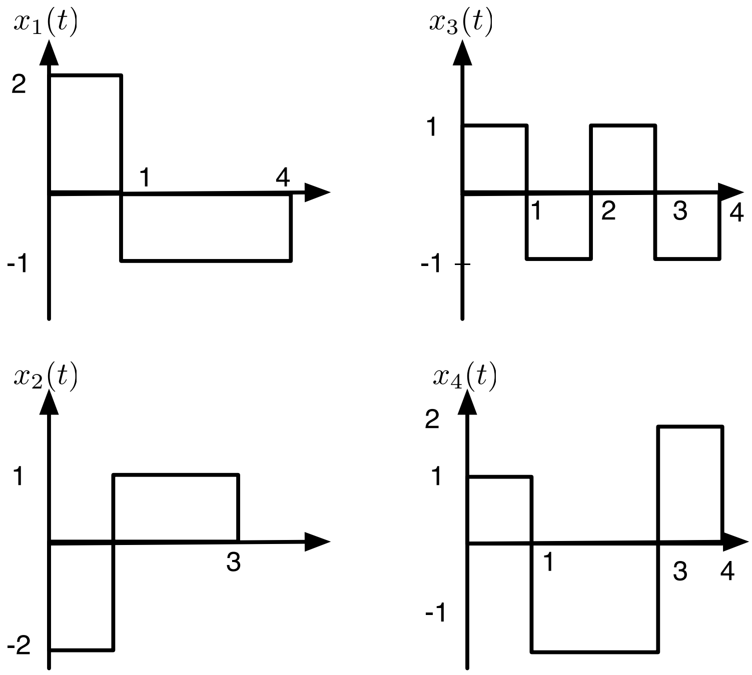


Figure 1: Signal set for Question 1.

(a) Determine the dimensionality of the waveforms and a set of orthonormal basis functions.

(b) Use the basis functions to represent the four waveforms by vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4$.

(c) Determine the minimum distance between any pair of vectors.

Question 2

Determine the signal space representation of the four signals $x_m(\cdot), m = 1, \ldots, 4$ shown in Figure 2 using as basis functions $f_1(\cdot)$ and $f_2(\cdot)$. Plot the signal space diagram and show that it is equivalent to that of QPSK modulation.

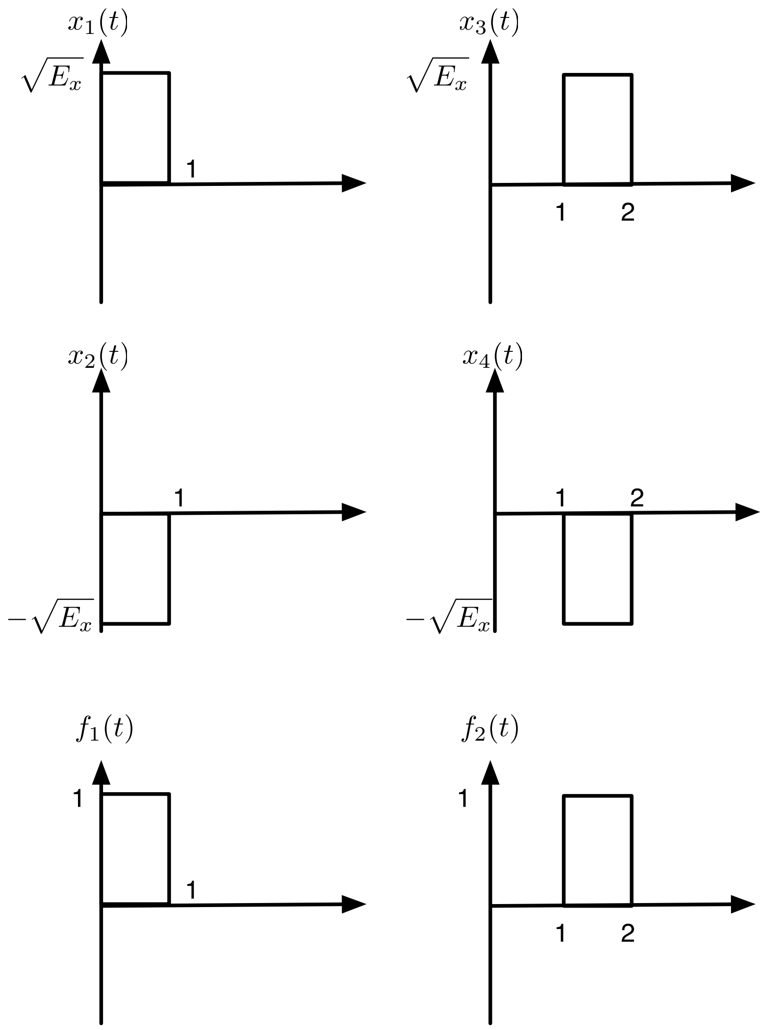


Figure 2: Signal set for Question 2.

Question 3

A binary digital communications system employs the signals

$$x_0(t) = 0, \quad t \in \mathbb{R}$$
$$x_1(t) = \begin{cases} A, & 0 \leq t \leq T \\ 0, & \text{otherwise} \end{cases}$$

for transmission of information. This is called *on-off* signalling.

(a) What are optimum receiver structures?

(b) Determine the optimum detector for the AWGN channel, assuming that the signals are equiprobable.

(c) Determine the probability of error as a function of SNR.

(d) How does on-off signalling compare with antipodal signalling, i.e., $x_0(t) = -A$, $x_1(t) = A$ for $0 \leq t \leq T$ and $x_0(t) = x_1(t) = 0$ otherwise?

Question 4

Suppose the energy limited signal $x(\cdot)$ is corrupted by the AWGN $n(\cdot)$. Hence, the observed signal is

$$y(t) = x(t) + n(t), \quad t \in \mathbb{R}.$$

The received signal is passed through a filter whose impulse response is $t \mapsto h(t)$. Find the filter $h(\cdot)$ that maximises the SNR at its output at $t = 0$.

Question 5

Suppose a signal set $x_m(\cdot), m = 1, \ldots, M$ is transmitted over an AWGN channel with noise variance $\sigma^2 = \frac{N_0}{2}$. Hence, the received signal is

$$y(t) = x_m(t) + n(t), \quad t \in \mathbb{R}.$$

A correlator receiver using the K signal-space basis functions $f_k(\cdot)$, $k = 1, \ldots, K$ is employed, producing

$$y_k = \langle y(t), f_k(t) \rangle = x_{m,k} + n_k, \quad k = 1, \ldots, K.$$

Show that

(a) $\mathbb{E}[n_k] = 0$.

(b) $\mathbb{E}[n_k n_l] = \frac{N_0}{2}\delta_{k,l}$, where $\delta_{k,l} = 1$ for $k = l$ and $\delta_{k,l} = 0$ otherwise.

(c) $\mathbb{E}[n'(t)y_k] = 0$, where $n'(t) = n(t) - \sum_{k=1}^{K} n_k f_k(t)$ is the AWGN component that does not lie within the signal space.

(d) Show that the optimum demodulator/detector can be obtained as shown in Figure 3, where $E_m = \|x_m(t)\|^2$, $m = 1, \ldots, M$.

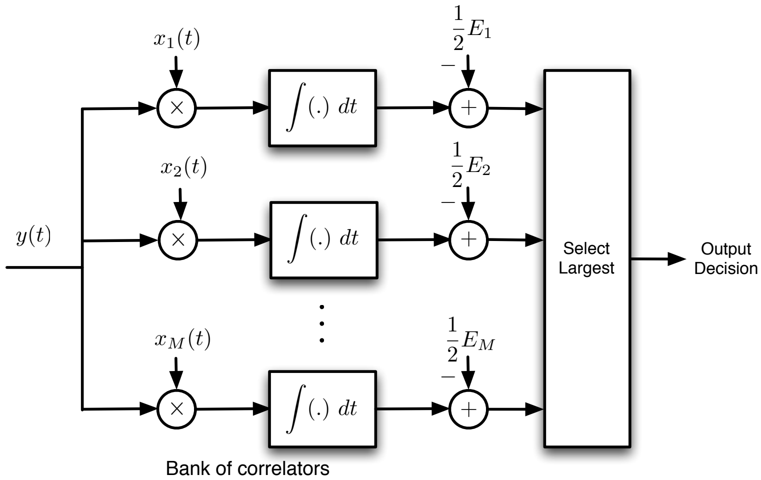


Figure 3: Detector for Question 5.

Question 6

Show the following results:

(a) Chain rule for entropy:

$$H(X, Y) = H(X) + H(Y|X).$$

(b) Positiveness:

$$I(X; Y) \geq 0.$$

(c) Conditioning reduces entropy:

$$H(X) \geq H(X|Y).$$

(d) Chain rule for mutual information:

$$I(X_1, X_2; Y) = I(X_1; Y) + I(X_2; Y|X_1).$$

(e) Data Processing Inequality: Let $X \to Y \to Z$ form a Markov chain. Show that

$$I(X; Z) \leq I(X; Y).$$

(f) Let $\boldsymbol{X}, \boldsymbol{Y}$ be n-dimensional random vectors over $\mathcal{X}^n, \mathcal{Y}^n$. Assume that

$$P_{\boldsymbol{Y}|\boldsymbol{X}}(\boldsymbol{y}|\boldsymbol{x}) = \prod_{i=1}^{n} P_{Y|X}(y_i|x_i).$$

Show that, for any distribution on $\boldsymbol{X}$,

$$I(\boldsymbol{X}; \boldsymbol{Y}) \leq \sum_{i=1}^{n} I(X_i; Y_i).$$

Question 7

Consider the cascade of channels shown in Figure 4. Show that the channel capacity of the cascade of channels cannot be larger than the channel capacity of the individual binary symmetric channels. Determine the channel capacity of the cascade of channels.

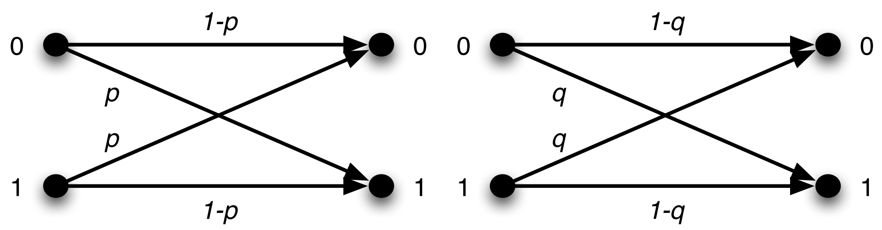
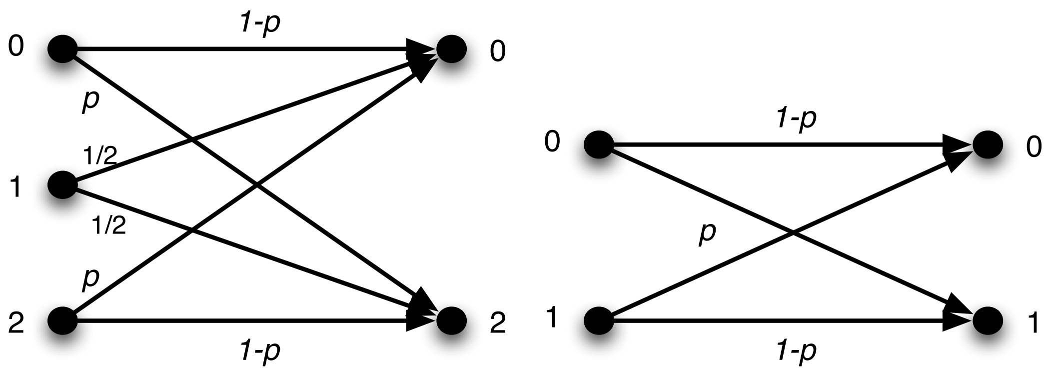


Figure 4: BSC channel cascade for Question 7.

Question 8

What is the channel capacity and the capacity achieving input distribution of the ternary-input binary-output channel given in Figure 5(a)? Compare this capacity to the channel capacity of the binary symmetric channel depicted in Figure 5(b). (Recall that the channel capacity of the binary symmetric channel with cross-over probability p is given by $C = 1 - H_b(p)$, where $H_b(\cdot)$ is the binary entropy function.)



(a) Ternary-input binary-output channel

(b) BSC

Figure 5: BSC and ternary-input binary-output channel for Question 8.

Question 9

Consider the three channels that are depicted in Figure 6.

(a) Find the capacity of channel 1. What input distribution achieves the capacity?

(b) Find the capacity of channel 2. What input distribution achieves the capacity?

(c) Let C_3 denote the capacity of the third channel, and let C_1 and C_2 denote the capacities of the first and second channel. Which of the following relations holds true and why?

(i) $C < \frac{1}{2}(C_1 + C_2)$
(ii) $C = \frac{1}{2}(C_1 + C_2)$
(iii) $C > \frac{1}{2}(C_1 + C_2)$

Hint: Use that

$$I\big(Q, \lambda W_1 + (1-\lambda)W_2\big) \geq \lambda I(Q, W_1) + (1-\lambda)W_2, \quad 0 < \lambda < 1$$

with equality if, and only if, $W_1 = W_2$. Here $I(Q, W)$ denotes the mutual information between X and Y if X is distributed according to Q and $P_{Y|X} = W$.

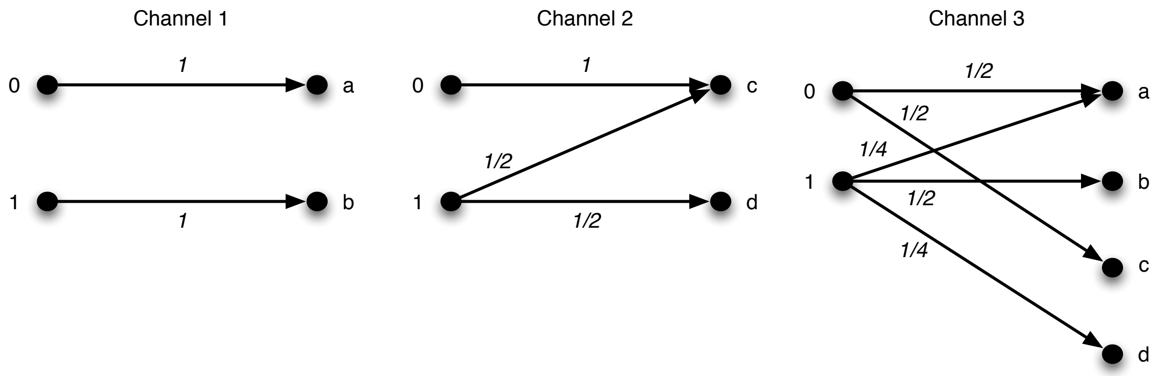


Figure 6: Channels for Question 9.

Question 10

Let C denote the capacity of a discrete memoryless channel with input alphabet $\mathcal{X} = \{x_1, \ldots, x_M\}$ and output alphabet $\mathcal{Y} = \{y_1, \ldots, y_N\}$. Show that $C \leq \min\{\log_2 M, \log_2 N\}$.

Question 11

(Converse to Fano's inequality). Suppose we want to estimate $X \in \mathcal{X}$ from the observation $Y \in \mathcal{Y}$, where $\mathcal{X}$ and $\mathcal{Y}$ are some finite sets. Fano's inequality states that, for any estimator $\hat{X} = g(Y)$ such that $X \to Y \to \hat{X}$ forms a Markov chain, we have that

$$H(X|Y) \leq H_b(P_e) + P_e \log_2(|\mathcal{X}| - 1)$$

where $P_e = \Pr(\hat{X} \neq X)$, and where $H_b(\cdot)$ is the binary entropy function. Using the graphical approach presented in the lecture notes, can you state a lower bound on $H(X|Y)$ in function of P_e to complement Fano's inequality? Try to provide a lower bound for the conditional entropy given a specific observation $H(X|Y = y)$ first, assuming $P_e(y) < 0.5$. Then consider what happens when $0.5 \leq P_e(y) < 2/3$, and generalise to $1 - \frac{1}{k} \leq P_e(y) < 1 - \frac{1}{k+1}$. Finally, note that if your bound on $H(X|Y = y)$ is not convex in $P_e(y)$, then you cannot use Jensen's inequality directly as was done in the lecture notes for the upper bound, and hence need to take a convex envelope of your bound to make the step from $H(X|Y = y)$ to $H(X|Y)$.

Question 12

(a) Suppose that X is a discrete random variable taking only nonnegative values. Show that for every $\delta > 0$

$$\Pr(X \geq \delta) \leq \frac{\mathbb{E}[X]}{\delta}.$$

This is called *Markov's inequality.*

(b) Use Markov's inequality to show that for any discrete random variable Y of variance σ_Y^2 we have for every $\delta > 0$

$$\Pr\left(\left(Y - \mathbb{E}[Y]\right)^2 \geq \delta\right) \leq \frac{\sigma_Y^2}{\delta}.$$

This is called *Chebyshev's inequality.*

(c) Use Markov's inequality to show that for every $\beta > 0$

$$\Pr\left(i(\bar{\boldsymbol{X}};\boldsymbol{Y}) > \log_2 \beta\right) \le \frac{1}{\beta}$$

where $i(x;y) \triangleq \log_2 \frac{P_{\boldsymbol{X},\boldsymbol{Y}}(\boldsymbol{x},\boldsymbol{y})}{P_{\boldsymbol{X}}(\boldsymbol{x})P_{\boldsymbol{Y}}(\boldsymbol{y})}$, and where $P_{\bar{\boldsymbol{X}},\boldsymbol{Y}}(\boldsymbol{x},\boldsymbol{y}) = P_{\boldsymbol{X}}(\boldsymbol{x})P_{\boldsymbol{Y}}(\boldsymbol{y})$.

(d) Use Chebyshev's inequality to prove the weak law of large numbers. Thus, show that for every $\epsilon > 0$

$$\lim_{n\to\infty} \Pr\left(\left|\frac{Z_1 + \ldots + Z_n}{n} - \mu\right| \ge \epsilon\right) = 0$$

where $Z_1, \ldots, Z_n$ is a sequence of i.i.d. random variables of mean μ and variance σ^2:

(i) Compute the mean and variance of the random variable

$$X_n \triangleq \frac{1}{n}\sum_{k=1}^{n} Z_k.$$

(ii) For every $n = 1, 2, \ldots$, upper-bound the probability

$$\Pr\left(\left|\frac{Z_1 + \ldots + Z_n}{n} - \mu\right| \ge \epsilon\right), \quad \epsilon > 0$$

using Chebyshev's inequality. Show that the weak law of large numbers follows from this upper bound by letting n tend to infinity.

Question 13

Consider the parallel discrete memoryless channels shown in Figure 7. Show that the capacity of the parallel channels is equal to the sum of the capacities of the individual channels. Thus, show that

$$C_{1,\ldots,K} = \max_{P_{X_1,\ldots,X_K}} I(X_1,\ldots,X_K;Y_1,\ldots,Y_K) = \sum_{k=1}^{K} C_k$$

where $C_k = \max\limits_{P_{X_k}} I(X_k;Y_k)$.

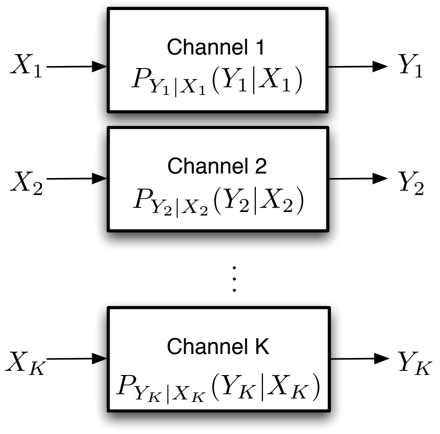


Figure 7: Channels for Question 13.