4F5: Advanced Wireless Communications

Handout 1: Introduction and Signal Space

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Lent Term 2012

Outline

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Course Organisation

Course Structure

- 9 chapters (some spanning multiple lectures)
- 2 examples papers
- Possibly one guest lecture (to be confirmed)
- Slide printouts handed out at the beginning of each chapter

Contact

- Questions, interruptions, or other active participation in class is encouraged!
- Website: `http://www-sigproc.eng.cam.ac.uk/~js851/teaching.html`
- All slides are on the website, but warning: slides may change based on feedback and other inputs from the class. Consult the latest version when revising.
- Email for questions or feedback: `jossy.sayir@eng.cam.ac.uk`

Acknowledgment

The slides are largely based on the course by Dr. Albert Guillén i Fàbregas given in previous years, with some inputs by Dr. Tobias Koch and by myself

Course Contents

Topics

- **Digital Modulation**
  - Signal Space
  - Optimum Signal Detection
  - Error Probability
- **Error-Control Coding**
  - *Review* of Channel Capacity
  - Block Codes
  - Error Probability
  - Convolutional Codes
  - Turbo-Codes
  - Low-Density Parity-Check Codes
  - Coded Modulation
- **Wireless Channels**
  - Multipath Propagation and Statistical Models
  - Equalisation and OFDM
  - Channel Capacity and Outage Probability
  - Diversity techniques

Recommended Books

- T. M. Cover and J. A. Thomas,
  Elements of Information Theory,
  Wiley Series in Telecommunications, 2nd Edition, 2006.
- D. J. C. MacKay,
  Information theory, inference, and learning algorithms,
  Cambridge University Press, 2003. (free online version)
- R. G. Gallager,
  Principles of Digital Communications,
  Cambridge University Press, 2008.
- A. Lapidoth,
  A Foundation in Digital Communication,
  Cambridge University Press, 2009.
- A. Goldsmith,
  Wireless Communications,
  Cambridge University Press 2005. (free online version)
- D. Tse and P. Vishwanath,
  Fundamentals of Wireless Communication,
  Cambridge University Press, 2005. (free online version)

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Definition: Communication

The process of delivering information from an information source to a destination (through a communications channel).

Quoting Shannon's 1948 Paper:

"The fundamental problem of communication is that of reproducing at one point exactly or approximately a message selected at another point."

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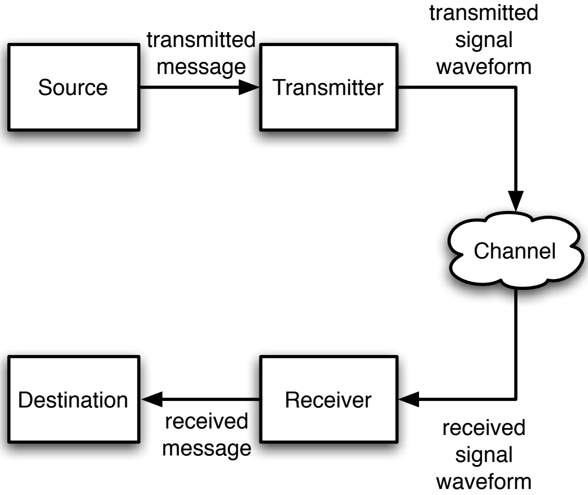
Basic Block Diagram



Basic Block Diagram

Component Description

- Source: voice, music, video (analogue), e-mail, file transfer (digital). Has an information message to transmit.
- Transmitter: translates the information message into a signal suitable for transmission over the channel.
- Channel: medium used to transmit the signal to the receiver: optical fibre, mobile wireless radio channel... Might add noise or interference.
- Receiver: reconstructs the message from the signal (inverse operation)
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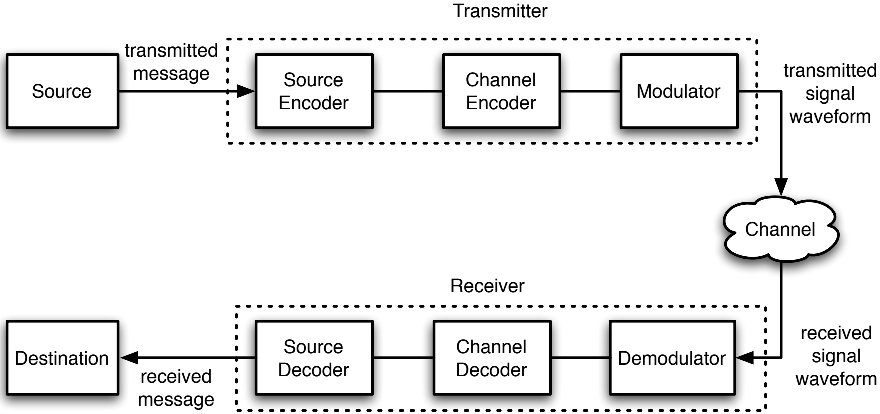
Basic Block Diagram

...with some more detail (digital communications)



Basic Block Diagram

Definition

- Source Encoder: lossless or lossy. Compresses the source message such that redundancy is removed. MP3, JPEG, MPEG are compression standards. (3F1)
- Channel Encoder: introduces *smart* redundancy suited to the channel characteristics (noise, interference...). (a bit in 3F4 and more in 4F5)
- Modulator: maps output of channel encoder to signal waveforms (electrical/optical signal), matched to the channel characteristics. (1B P6, 3F4 and 4F5)

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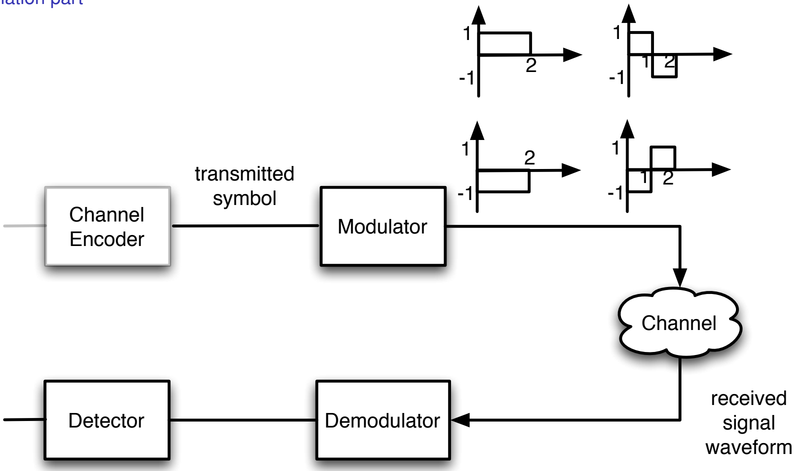
...modulation part



The Signal Space

Signal Space Concepts

- Let $\mathcal{L}_2$ be the set of complex-valued signals (functions) $x(.)$ with finite energy, i.e.,

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt < \infty.$$

  It can be shown that $\mathcal{L}_2$ is a vector space.

- The inner product between $x_1(.)$ and $x_2(.)$ in $\mathcal{L}_2$ is defined as

$$\langle x_1(.), x_2(.)\rangle \triangleq \int_{-\infty}^{\infty} x_1(t)x_2^*(t)dt$$

- The norm of a signal $x(.) \in \mathcal{L}_2$ is

$$\|x(.)\| = \sqrt{\langle x(t), x(t)\rangle} = \sqrt{\int_{-\infty}^{\infty} |x(t)|^2 dt}$$

- **Triangle inequality:** $\|x_1(.) + x_2(.)\| \leq \|x_1(.)\| + \|x_2(.)\|$
- **Cauchy-Schwarz inequality:**

$$|\langle x_1(.), x_2(.)\rangle| \leq \|x_1(.)\| \, \|x_2(.)\|$$

  with equality when $x_2(t) = ax_1(t)$, $t \in \mathbb{R}$ for $a \in \mathbb{C}$.

The Signal Space

Orthogonal Expansions of Signals

- The set of K signals $f_k(.), k = 1, \ldots, K$ is said to be orthonormal if

$$\langle f_n(.), f_m(.) \rangle = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$$

- Let $\mathcal{U}$ be a sub-space of $\mathcal{L}_2$. If $\mathcal{U}$ is of dimension K, then we can find an orthonormal basis $f_k(.), k = 1, \ldots, K$ such that for any $x(.) \in \mathcal{U}$, the signal

$$\hat{x}(.) = \sum_{k=1}^{K} c_k f_k(.), \quad c_k = \langle x(.), f_k(.) \rangle \text{ for } k = 1, \ldots, K \qquad (1)$$

satisfies

$$\|x(.) - \hat{x}(.)\| = 0 \quad \text{and} \quad \|x(.)\|^2 = \|\hat{x}(.)\|^2 = \sum_{k=1}^{K} c_k^2.$$

- If $\mathcal{U}$ has dimension larger than K, then we can approximate any $x(.) \in \mathcal{U}$ by $\hat{x}(.)$ defined as in (1) with error $e(.) = x(.) - \hat{x}(.)$ and error energy $E_e = \|e(.)\|^2$
- Since $\langle e(.), f_k(.) \rangle = 0$ for $k = 1, \ldots, K$, the orthogonality principle of Minimum Mean Squared Error (MMSE) estimation guarantees that $\hat{x}(.)$ is the approximation of $x(.)$ minimising E_e among all signals in the sub-space of $\mathcal{U}$ spanned by the basis $f_k(.), k = 1, \ldots, K$.

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Orthogonal Expansions of Signals

- For any M signals $x_m(.), m = 1, \ldots, M$, we can construct a set of $K \leq M$ orthonormal signals using the Gram-Schmidt procedure, for $m = 1, \ldots, M$:

$$f'_m(.) = x_m(.) - \sum_{i=1}^{m-1} \langle x_m(.), f_i(.) \rangle f_i(.),$$

  and

$$f_m(.) = \frac{f'_m(.)}{\|f'_m(.)\|} \quad \text{if} \quad \|f'_m(.)\| > 0,$$

  discarding all $f'_m(.)$ such that $\|f'_m(.)\| = 0$.
- $K = M$ iff all original M signals are linearly independent.
- Once we have constructed the orthonormal signal set $f_k(.), k = 1, \ldots, K$, we can express the set of original signals as vectors

$$\boldsymbol{x}_m = (x_{m,1}, \ldots, x_{m,K})$$

  where $x_{m,k} = \langle x_m(.), f_k(.) \rangle$ and

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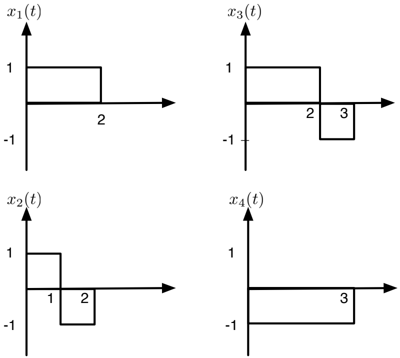
The Signal Space

Example

Consider the set of $M = 4$ signals given below and its corresponding orthonormal basis obtained with Gram-Schmidt: the dimension of the signal set is $K = 3$, and the vector representation is

$$\boldsymbol{x}_1 = (\sqrt{2}, 0, 0),\ \boldsymbol{x}_2 = (0, \sqrt{2}, 0),\ \boldsymbol{x}_3 = (\sqrt{2}, 0, 1),\ \boldsymbol{x}_4 = (-\sqrt{2}, 0, 1).$$

Note, that there is more than one orthonormal basis. Hence, there is more than one vector representation.



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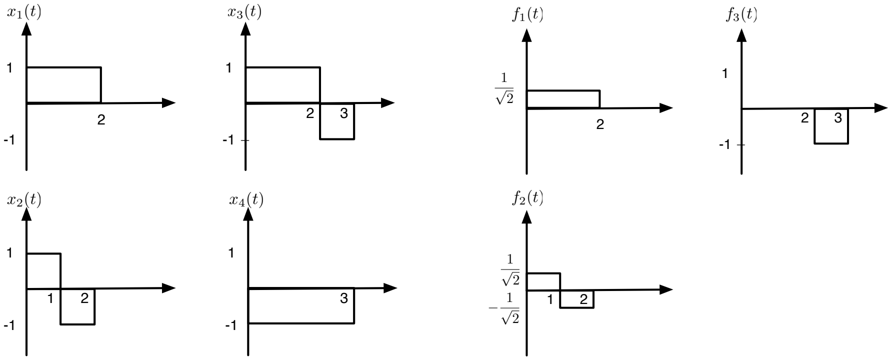


Outline

Digital Modulation: Phase-Shift Keying (PSK)

- For any real-valued pulse shape $g(.)$, the M signals can be expressed as

$$x_m(t) = g(t)\cos\left(2\pi f_c t + 2\pi\frac{m-1}{M}\right) = \mathrm{Re}\left\{g(t)e^{j2\pi(m-1)/M}\ e^{j2\pi f_c t}\right\}$$

$$= g(t)\cos\left(2\pi\frac{m-1}{M}\right)\cos(2\pi f_c t) - g(t)\sin\left(2\pi\frac{m-1}{M}\right)\sin(2\pi f_c t)$$

- For all m, the signal $x_m(.)$ has energy $\|x_m(.)\|^2 = \frac{\|g(.)\|^2}{2} = \frac{E_g}{2}$
- Basis functions $f_1(t) = \sqrt{\frac{2}{E_g}}g(t)\cos(2\pi f_c t)$ and $f_2(t) = -\sqrt{\frac{2}{E_g}}g(t)\sin(2\pi f_c t)$

  (Note that if $f_c >$ bandwidth of $g(.)$, then $f_1(.)$ and $f_2(.)$ are orthonormal.)
- Vector representation $\boldsymbol{x}_m = \left(\sqrt{\frac{E_g}{2}}\cos\left(2\pi\frac{m-1}{M}\right), \sqrt{\frac{E_g}{2}}\sin\left(2\pi\frac{m-1}{M}\right)\right)$
- Euclidean distance $d_{m,n} = \|\boldsymbol{x}_m - \boldsymbol{x}_n\|$. The minimum Euclidean distance $d_{\min} = \sqrt{E_g\left(1-\cos\frac{2\pi}{M}\right)}$

The Signal Space

Digital Modulation: Quadrature-Amplitude Modulation (QAM)

- For any real-valued pulse shape $g(.)$, the M signals can be expressed as

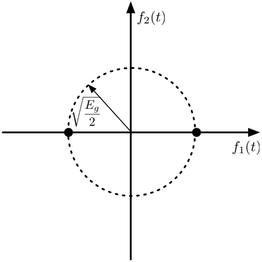
$$\begin{aligned} x_m(t) &= A_m^{(I)}\, g(t) \cos(2\pi f_c t) - A_m^{(Q)}\, g(t) \sin(2\pi f_c t) \\ &= \mathrm{Re}\left\{ \left(A_m^{(I)} + jA_m^{(Q)} \right) g(t) e^{j2\pi f_c t} \right\} \end{aligned}$$

  where $A_m^{(I)}, A_m^{(Q)}$ are the information-bearing signal amplitudes of the quadrature carriers. When $\log_2 M$ is even, it can be seen as 2 orthogonal PAM signals.
- Energy of $x_m(.)$ depends on m.
- Basis functions same as for PSK:

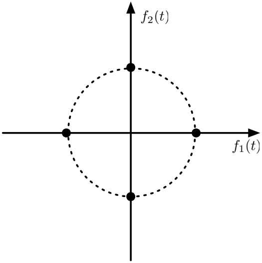
$$f_1(t) = \sqrt{\frac{2}{E_g}} g(t) \cos(2\pi f_c t) \quad \text{and} \quad f_2(t) = -\sqrt{\frac{2}{E_g}} g(t) \sin(2\pi f_c t)$$

- Vector representation $\boldsymbol{x}_m = \left(\sqrt{\frac{E_g}{2}} A_{c,m}, \sqrt{\frac{E_g}{2}} A_{s,m} \right)$
- For rectangular QAM with amplitudes $\{(2m-1-M)d, m = 1, \ldots, M\}$ the minimum Euclidean distance is $d_{\min} = d\sqrt{2E_g}$

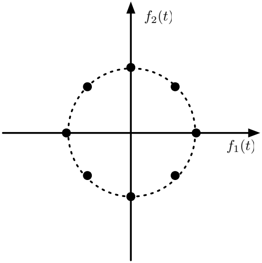
The Signal Space



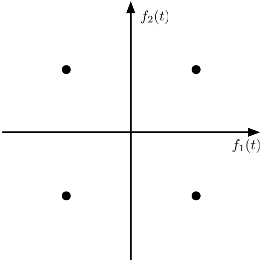
BPSK



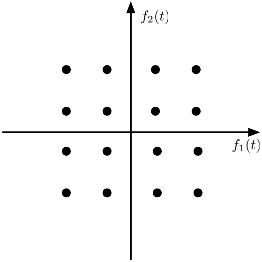
QPSK



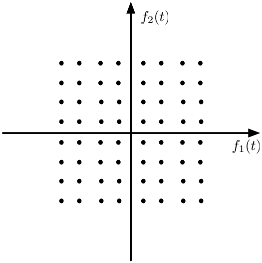
8-PSK



4-QAM



16-QAM



64-QAM

The Signal Space

Digital Modulation: Transmitting a Sequence of Symbols

- Transmitting only M signals with a signal $x_m(.)$ of infinite duration is not efficient.
- Transmit time-shifts of the same signal:

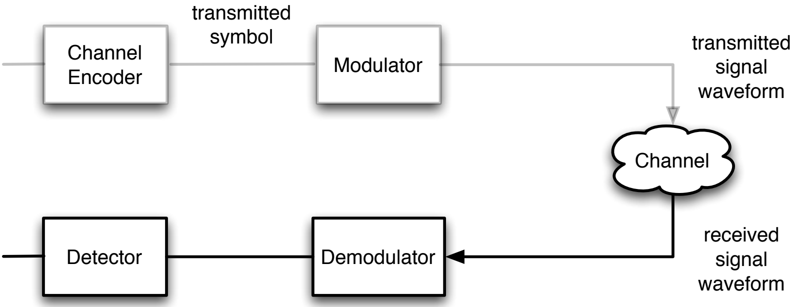
$$x_{\mathrm{PSK}}(t) = \sum_{\ell} g(t - \ell T)\left[\cos\left(2\pi \frac{m_\ell - 1}{M}\right) cos(2\pi f_c t) - \sin\left(\frac{m_\ell - 1}{M}\right)\sin(2\pi f_c t)\right]$$

$$x_{\mathrm{QAM}}(t) = \sum_{\ell} g(t - \ell T)\left[A^{(I)}_{m_\ell}\cos(2\pi f_c t) - A^{(Q)}_{m_\ell}\sin(2\pi f_c t)\right]$$

where $\{m_\ell\}$ is the sequence of symbols we would like to transmit.

Basic Block Diagram

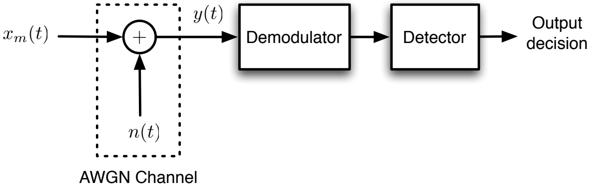
...modulation part



Outline

Optimum Demodulation and Detection

- M signals $x_m(.)$, $m = 1, \ldots, M$, contain $\log_2 M$ information bits if picked uniformly. Transmission over Additive White Gaussian Noise (AWGN) channel, i.e., the received signal is given by
$$y(.) = x_m(.) + n(.)$$
where $n(.)$ is AWGN of power spectral density $\Phi_{nn}(f) = \frac{N_0}{2}$ W/Hz.
- Optimum receiver (in the sense that it minimises the probability of making an error) consists of two parts: a demodulator and a detector.
- Demodulator projects received signal onto the signal space basis. Two implementations
  - correlation
  - matched filter
- Detector: makes a decision on the transmitted signal based on a given metric



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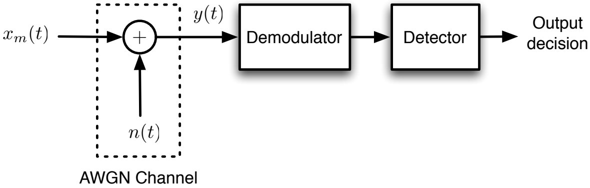

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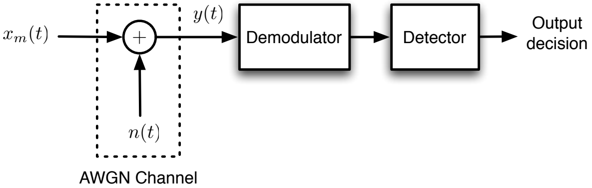

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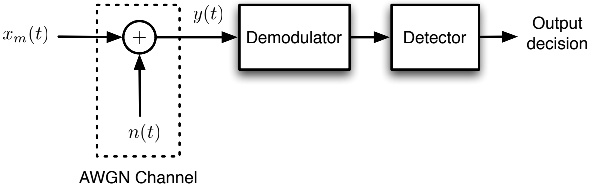
- Detector: makes a decision on the transmitted signal based on a given metric

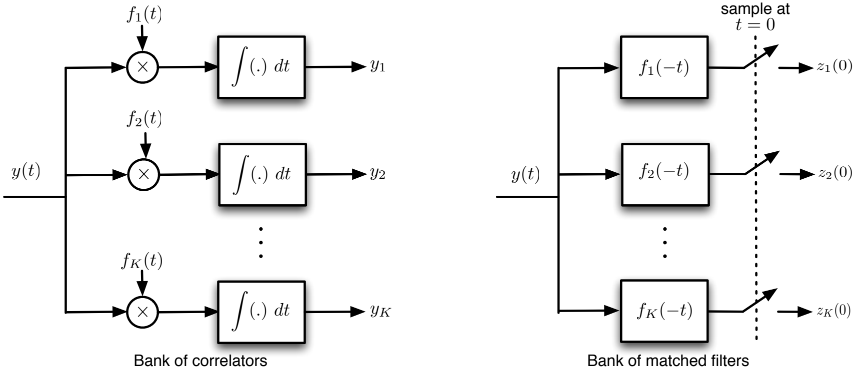


Optimum Demodulation

- Let $x_m(.) = \sum_{k=1}^{K} x_{m,k} f_k(.)$ where $f_k(.), k = 1, \ldots, K$ forms an orthonormal basis.
- If there were no noise, we could recover $x_{m,k}$ directly by computing
$$\langle y(.), f_k(.)\rangle = \left\langle \sum_{k'=1}^{K} x_{m,k'} f_{k'}(.), f_k(.) \right\rangle = \sum_{k'=1}^{K} x_{m,k'} \langle f_{k'}(.), f_k(.)\rangle = x_{m,k}$$
- With noise:
$$\langle y(.), f_k(.)\rangle = \left\langle \sum_{k'=1}^{K} x_{m,k'} f_{k'}(.) + n(.), f_k(.) \right\rangle = x_{k,m} + n_k, \quad k = 1, \ldots, K$$
where $x_{k,m} = \langle x_m(.), f_k(.)\rangle$ is known and $n_k = \langle n(.), f_k(.)\rangle$ is Gaussian noise.
- Bank of K cross-correlators: projects $y(.)$ onto the signal space
$$y_k \triangleq \int_{-\infty}^{\infty} y(t) f_k(t) dt = \langle y(.), f_k(.)\rangle, \quad k = 1, \ldots, K$$
- Bank of K matched filters with responses $h_k(t) = f_k(-t)$
$$z_k(t) \triangleq \int_{-\infty}^{\infty} y(\tau) h_k(t-\tau) d\tau = \int_0^t y(\tau) f_k(\tau - t) d\tau, \quad k = 1, \ldots, K$$
Sampling at $t = 0$ yields $z_k(0) = \langle y(.), f_k(.)\rangle$.

The Signal Space

...bank of correlators or matched filters



The Signal Space

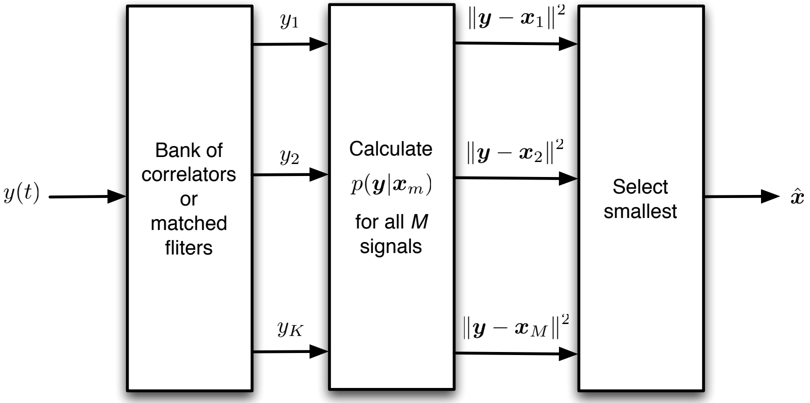
Optimum Demodulation: Noise Properties

- Approximating $y(.)$ as a linear combination of $f_k(.), k = 1, \ldots, K$ yields
$$y(.) = \sum_{k=1}^{K} y_k f_k(.) + n'(.) = \sum_{k=1}^{K} x_{m,k} f_k(.) + \sum_{k=1}^{K} n_k f_k(.) + n'(.)$$
where $n'(.) = n(.) - \sum_{k=1}^{K} n_k f_k(.)$ is the part of the AWGN that is not projected onto the signal space.
- $n_k, k = 1, \ldots, K$ are uncorrelated Gaussian RVs of mean 0 and variance $\sigma^2 = \frac{N_0}{2}$
$$\mathbb{E}[n_k] = \int_{-\infty}^{\infty} \mathbb{E}[n(t)] f_k(t) dt = 0$$
$$\mathbb{E}[n_k n_l] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbb{E}[n(t)n(\tau)] f_k(t) f_l(\tau) dt\, d\tau = \frac{N_0}{2} \int_{-\infty}^{\infty} f_k(t) f_l(t) dt = \frac{N_0}{2} \delta[k-l]$$
- Conditioned on $x_{m,k}$, the noise term $n'(.)$ and the observations $y_k, k = 1, \ldots, K$ are uncorrelated.
- Since, conditioned on $\boldsymbol{x}_m = (x_{m,1}, \ldots, x_{m,K})$, $\boldsymbol{y} = (y_1, \ldots, y_K)$ and $n'(.)$ are uncorrelated Gaussians (and are therefore independent), and since $n'(.)$ does not depend on $\boldsymbol{x}_{m,k}$, it follows that $n'(.)$ is irrelevant for making decisions. Thus, only the noise projected onto the signal space is relevant for making decisions.

Optimum Detection

- Decisions are made based on
$$y_k = x_{m,k} + n_k, \quad k = 1, \ldots, K$$
- Since $n_k \sim \mathcal{N}(0, \frac{N_0}{2})$, we have $p(y_k|x_{m,k}) = \frac{1}{\sqrt{\pi N_0}} e^{-\frac{1}{N_0}(y_k - x_{m,k})^2}$
- Since the noise components are independent, we have that for $m = 1, \ldots, M$
$$p(\boldsymbol{y}|\boldsymbol{x}_m) = \frac{1}{(\pi N_0)^{K/2}} e^{-\frac{1}{N_0}\sum_{k=1}^{K}(y_k - x_{m,k})^2} = \frac{1}{(\pi N_0)^{K/2}} e^{-\frac{1}{N_0}\|\boldsymbol{y}-\boldsymbol{x}_m\|^2}$$
- Given $\boldsymbol{y}$, the optimum detector decides on the $\boldsymbol{x}_m$ that maximises the probability of a correct decision.
- Let $p(\boldsymbol{x}_m|\boldsymbol{y}) \triangleq \Pr\{\text{signal } \boldsymbol{x}_m \text{ was transmitted} \mid \boldsymbol{y} \text{ was received}\}$. The optimum detector selects the $\boldsymbol{x}_m$ that maximizes the a posteriori probability (MAP)
$$\hat{\boldsymbol{x}} = \arg \max_{m=1,\ldots,M} p(\boldsymbol{x}_m|\boldsymbol{y}) = \arg \max_{m=1,\ldots,M} \frac{p(\boldsymbol{y}|\boldsymbol{x}_m)p(\boldsymbol{x}_m)}{p(\boldsymbol{y})} = \arg \max_{m=1,\ldots,M} p(\boldsymbol{y}|\boldsymbol{x}_m)p(\boldsymbol{x}_m)$$
- If all signals are equally likely, i.e., $p(\boldsymbol{x}_m) = 1/M$, for $m = 1, \ldots, M$, then
$$\hat{\boldsymbol{x}} = \arg \max_{m=1,\ldots,M} p(\boldsymbol{y}|\boldsymbol{x}_m) = \arg \max_{m=1,\ldots,M} \log p(\boldsymbol{y}|\boldsymbol{x}_m) = \arg \min_{m=1,\ldots,M} \|\boldsymbol{y} - \boldsymbol{x}_m\|^2$$

The Signal Space



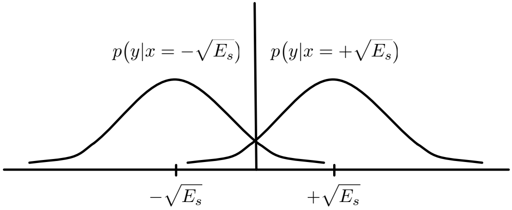
Outline

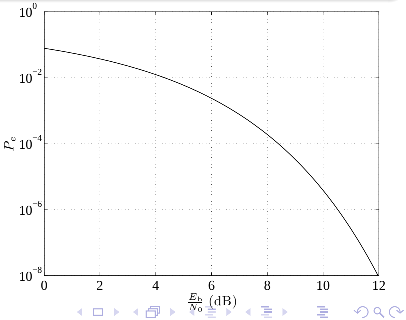
Error Probability

- BPSK: $K = 1$ and $x \in \{-\sqrt{E_s}, +\sqrt{E_s}\}$, with $E_s = \|g(.)\|^2$ the symbol energy.
- Let E_b denote the energy per bit. For BPSK we have $E_b = E_s$. In general, $E_s = R\,E_b$, where R is the rate in bits/symbol.
- The error probability is given by

$$P_e = \frac{1}{2}\Pr\{\text{error}|x = +\sqrt{E_s}\} + \frac{1}{2}\Pr\{\text{error}|x = -\sqrt{E_s}\} = \Pr\{\text{error}|x = +\sqrt{E_s}\}$$

$$= \int_{-\infty}^{0} p(y|x = +\sqrt{E_s})\,dy = \int_{-\infty}^{0} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{1}{N_0}(y-\sqrt{E_s})^2}\,dy = Q\left(\sqrt{2\frac{E_s}{N_0}}\right)$$





The Signal Space

Error Probability

- PSK: in general, for $M > 4$ we need numerical integration. When $M = 4$ we have 2 orthogonal BPSK in phase and quadrature,

$$P_e = 1 - P_c = 1 - \left(1 - P_e^{\text{bpsk}}\right)^2 = 1 - \left(1 - Q\left(\sqrt{2\frac{E_b}{N_0}}\right)\right)^2$$

- QAM: when $M = M'^2$, we have 2 orthogonal M'-PAM in phase and quadrature,

$$P_e = 1 - P_c = 1 - \left(1 - P_e^{\text{pam}}\right)^2$$

$$P_e^{\text{pam}} = 2\left(1 - \frac{1}{M'}\right) Q\left(\sqrt{\frac{3}{M-1}\frac{E_s}{N_0}}\right)$$

- Union bound: simple and accurate (at large SNR) bound to the error probability

$$P_e = \frac{1}{M}\sum_{m=1}^{M} \Pr\{\text{error}|\boldsymbol{x}_m \text{ was transmitted}\} = \frac{1}{M}\sum_{m=1}^{M} \Pr\left\{\bigcup_{m' \neq m} \{\hat{\boldsymbol{x}} = \boldsymbol{x}_{m'}\} \middle| \boldsymbol{x}_m\right\}$$

$$\leq \frac{1}{M}\sum_{m=1}^{M}\sum_{m' \neq m} \Pr\{\hat{\boldsymbol{x}} = \boldsymbol{x}_{m'}|\boldsymbol{x}_m\} = \frac{1}{M}\sum_{m=1}^{M}\sum_{m' \neq m} Q\left(\sqrt{\frac{\|\boldsymbol{x}_m - \boldsymbol{x}_{m'}\|^2}{2N_0}}\right)$$

The Signal Space

Computer Exercise

- Simulate the error probability vs SNR curve of BPSK and compare to theory
- Simulate the error probability of 16-QAM and compare to theory and union bound
- for each SNR value
  - Generate at random symbols from the constellation
  - Generate and add noise with given noise variance according to SNR
  - Calculate metrics given the received signal
  - Decide in favour of the largest
  - Check with transmitted symbol and count errors $P_e \approx \frac{\text{number of errors counted}}{\text{number of transmitted symbols}}$
- end for