4F5: Advanced Wireless Communications

Handout 2: Review of Channel Capacity

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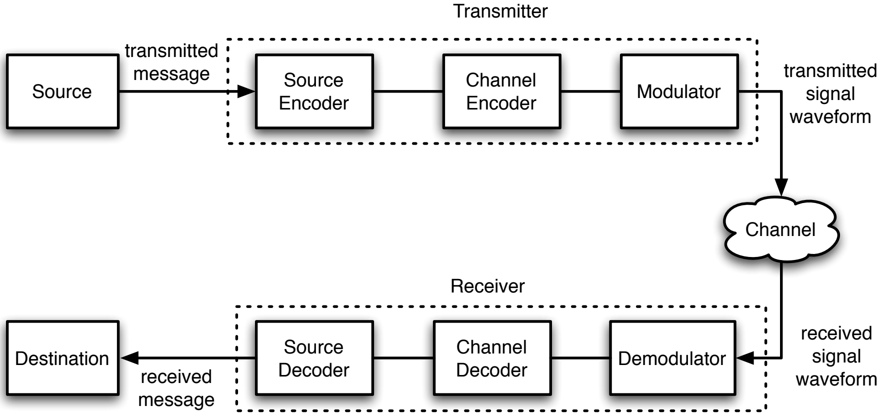
Lent 2012

Reminder: Basic Block Diagram

...with some more detail (digital communications)



Information Theoretic Channel

Definition (Kelly)

A *channel* is that part of the communication system that one is either unwilling or unable to change.

Outline

Definitions

Entropy, Divergence

Let the random variables X, Y take value in the sets $\mathcal{X}, \mathcal{Y}$. We define (in bits)

- Entropy / Uncertainty

$$H(X) = H(P_X) \stackrel{\text{def}}{=} -\sum_{x \in \mathcal{X}} P_X(x) \log_2 P_X(x) = -\mathbb{E}[\log_2 P_X(X)]$$

- Divergence / Relative Entropy / Kullback-Leibler "Distance"

$$D(P_X \| Q_X) \stackrel{\text{def}}{=} \sum_{x \in \mathcal{X}} P_X(x) \log_2 \frac{P_X(x)}{Q_X(x)} = \mathbb{E}\left[\log_2 \frac{P_X(X)}{Q_X(X)}\right]$$

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$$H(X, Y) \stackrel{\text{def}}{=} H(P_{XY}) = -\mathbb{E}[\log_2 P_{X,Y}(X, Y)]$$

- Conditional Entropy (conditioned on an event)

$$H(X|Y = y) \stackrel{\text{def}}{=} H(P_{X|Y=y}) = -\mathbb{E}\left[\log_2 P_{X|Y}(X|y)\right]$$

- Conditional Entropy/Equivocation

$$H(X|Y) \stackrel{\text{def}}{=} \sum_y P_Y(y) H(X|Y = y) = -\mathbb{E}[\log_2 P_{X|Y}(X|Y)]$$

- Mutual Information

$$I(X; Y) \stackrel{\text{def}}{=} H(X) - H(X|Y) = H(Y) - H(Y|X)$$
$$= D(P_{XY} \| P_X P_Y) = \mathbb{E}\left[\log_2 \frac{P_{X,Y}(X, Y)}{P_X(X) P_Y(Y)}\right]$$

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Properties

Properties of Entropy, Mutual Information and Relative Entropy

1. Chain rules

$$H(X, Y) = H(X) + H(Y|X)$$
$$I(X_1, X_2; Y) = I(X_1; Y) + I(X_2; Y|X_1)$$

where $I(X_2; Y|X_1) \stackrel{\text{def}}{=} H(X_2|X_1) - H(X_2|X_1 Y)$.

2. Positiveness

$$\textbf{entropy:} \qquad H(X) \geq 0, \quad \text{with equality iff } X \text{ is deterministic}$$

implies positiveness of conditional entropy.

$$\textbf{relative entropy:} \qquad D(P_X||Q_X) \geq 0, \quad \text{with equality iff } P_X = Q_X$$

implies $I(X; Y) \geq 0$ (equality iff X and Y are independent).

3. Conditioning reduces entropy

$$H(X|Y) \leq H(X)$$

4. Maximum entropy

$$H(X) \leq \log |\mathcal{X}|, \quad \text{with equality iff } X \text{ is uniform}$$

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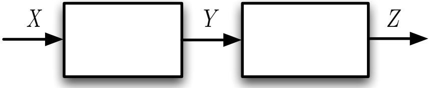
Properties

Data Processing Inequality

Let $X \rightarrow Y \rightarrow Z$ form a Markov chain (i.e., $P_{XZ|Y} = P_{X|Y}P_{Z|Y}$). Then

$$I(X; Z) \leq I(X; Y)$$



Channel Coding

Channel Definitions

- Channel input X over alphabet $\mathcal{X}$.
- Channel output Y over alphabet $\mathcal{Y}$.
- Sequence of transition probabilities

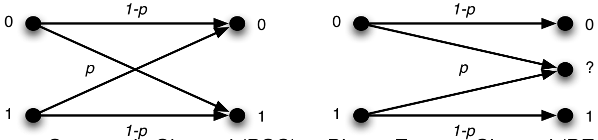
$$\left\{P_{\boldsymbol{Y}|\boldsymbol{X}}(y_1,\ldots,y_n|x_1,\ldots,x_n)\ :\ n=1,2,\ldots\right\}$$

- Memoryless channel: for $\boldsymbol{x}\in\mathcal{X}^n, \boldsymbol{y}\in\mathcal{Y}^n$

$$P_{\boldsymbol{Y}|\boldsymbol{X}}(\boldsymbol{y}|\boldsymbol{x})=\prod_{i=1}^{n}P_{Y|X}(y_i|x_i)$$

- Discrete Memoryless Channel ($|\mathcal{X}|,|\mathcal{Y}|<\infty$) defined by transition matrix $\boldsymbol{P}$

$$[\boldsymbol{P}]_{i,j}=\Pr(Y=i|X=j)$$



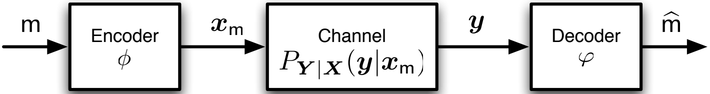
Binary Symmetric Channel (BSC) Binary Erasure Channel (BEC)

Channel Coding

Channel Coding Definitions

A channel coding scheme, or *block code*, is defined by

- A codebook $\mathcal{C} \subseteq \mathcal{X}^n$;
- a uniformly distributed message $\mathrm{m} \in \mathcal{M} = \{1, \ldots, |\mathcal{M}|\}$ (note that $|\mathcal{C}| \leq |\mathcal{M}|$);
- the sequences $\boldsymbol{x} \in \mathcal{C}$ are called codewords;
- the coding rate $R = \frac{1}{n} \log_2 |\mathcal{M}|$ (bits/channel use);
- an encoding function $\phi : \mathcal{M} \to \mathcal{C}$ such that $\phi(\mathrm{m}) = \boldsymbol{x}_\mathrm{m}$ is the codeword corresponding to message $\mathrm{m} \in \mathcal{M}$;
- a decoding function $\varphi : \mathcal{Y}^n \to \mathcal{M}$ such that $\varphi(\boldsymbol{y}) = \hat{\mathrm{m}}$ maps the received sequence to an estimated information message.



Channel Coding

Example

Consider a binary ($\mathcal{X} = \{0, 1\}$) code $\mathcal{C}$ of length $n = 4$ defined as

$$\mathcal{C} = \{(0,0,0,0), (0,0,1,1), (1,1,0,0), (1,1,1,1)\}$$

The rate of the code is $R = \frac{1}{4} \log_2 |\mathcal{M}| = \frac{1}{4} \log_2 4 = \frac{1}{2}$. The message set $\mathcal{M} = \{1, 2, 3, 4\}$ can be represented by 2 bits. Hence the encoder has as input 2 bits and outputs 4 bits (adds redundancy).

Error Probability

The average message (or codeword) error probability of the code $\mathcal{C}$ is defined as

$$P_e \triangleq \frac{1}{|\mathcal{M}|} \sum_{\mathrm{m}=1}^{|\mathcal{M}|} P_e(\mathrm{m}) = \frac{1}{|\mathcal{M}|} \sum_{\mathrm{m}=1}^{|\mathcal{M}|} \sum_{\boldsymbol{y}:\varphi(\boldsymbol{y}) \neq \mathrm{m}} P_{\boldsymbol{Y}|\boldsymbol{X}}(\boldsymbol{y}|\boldsymbol{x}_{\mathrm{m}} = \phi(\mathrm{m}))$$

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Channel Coding

Achievable Rates and Capacity

- A rate R is said to be achievable if there exist codes $\mathcal{C}$ of length n equipped with encoding and decoding functions ϕ, φ such that, for every $\epsilon > 0$ and every $n \geq n_\epsilon$ (for some n_ϵ),
$$\frac{1}{n}\log_2 |\mathcal{M}| \geq R \quad \text{and} \quad P_e \leq \epsilon$$
- The channel capacity C is defined as the supremum of all achievable rates.
- Thus, for transmission rates $R < C$ there exist coding schemes with arbitrarily small error probability (for sufficiently large block length), while for $R > C$ there exist no such schemes.

Theorem (Shannon's noisy channel coding theorem)

The channel capacity for a memoryless channel $P_{Y|X}(\cdot)$ is given by
$$C = \max_{P_X(\cdot)} I(X;Y)$$
where the maximisation is over all probability distributions on the channel input X.

Channel Capacity

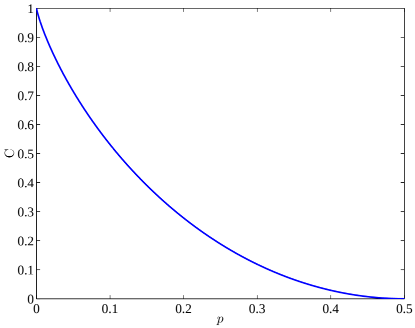
Example

- BSC

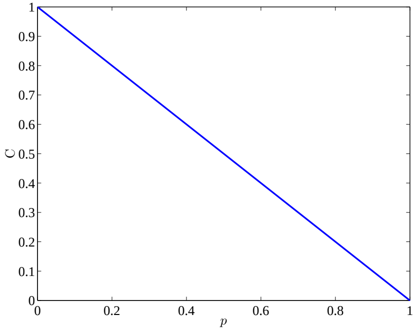
$$C = 1 - H_b(p), \quad P_X^\star(0) = P_X^\star(1) = \frac{1}{2}$$

- BEC

$$C = 1 - p, \quad P_X^\star(0) = P_X^\star(1) = \frac{1}{2}$$



BSC



BEC

Channel Capacity

Example (AWGN Channel)

- AWGN channel with noise power σ^2 and input power constraint P, i.e.,

$$P_{Y|X}(y|x) = \frac{1}{\pi\sigma^2} e^{-\frac{|y-x|^2}{\sigma^2}}, \quad x, y \in \mathbb{C} \quad \text{and} \quad \mathbb{E}[|X|^2] \leq P$$

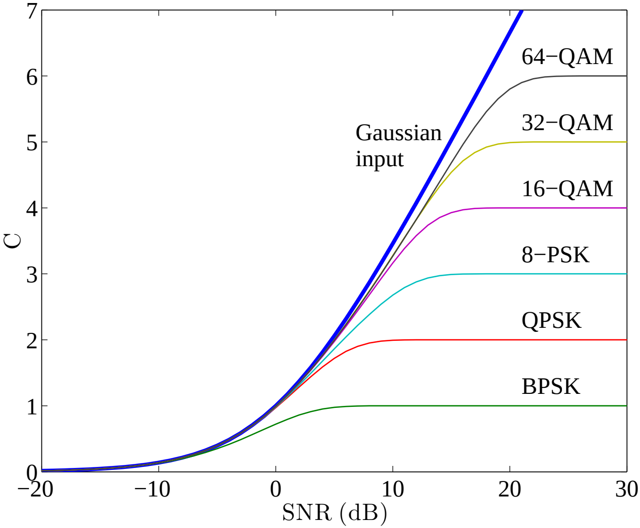
- Capacity is

$$C = \log_2(1 + \mathrm{SNR}), \quad \mathrm{SNR} = \frac{P}{\sigma^2} \qquad P_X^\star(x) = \frac{1}{\pi P} e^{-\frac{|x|^2}{P}}$$

- Gaussian inputs are not practical; we commonly resort to modulations such as PSK/QAM, assuming $P_X(x) = \frac{1}{|\mathcal{X}|}$, $x \in \mathcal{X}$:

$$I(X;Y) = \frac{1}{|\mathcal{X}|} \sum_{x \in \mathcal{X}} \mathbb{E}\left[\log_2 \frac{P_{Y|X}(Y|x)}{\frac{1}{|\mathcal{X}|} \sum_{x' \in \mathcal{X}} P_{Y|X}(Y|x')}\right]$$

Channel Capacity
AWGN Channel



Channel Capacity

Computer Exercise

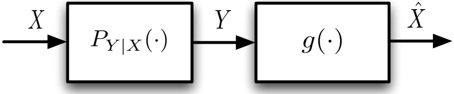
- Simulate the mutual information curves for BPSK, QPSK, 8-PSK and 16-QAM.
- Let $P_{Y|X}(y|x) = \frac{1}{\pi\sigma^2} e^{-\frac{1}{\sigma^2}|y-x|^2}$, $x, y \in \mathbb{C}$.
- For each SNR value
  - calculate the expectation over X as $\frac{1}{M}\sum_{x\in\mathcal{X}}$
  - calculate the expectation over $Y|X$ by randomly generating noise samples $\sim \mathcal{N}_{\mathbb{C}}(0, \sigma^2)$ and average Montecarlo
- end for
- TIP: Normalise your constellations in energy, i.e., $\frac{1}{|\mathcal{X}|}\sum_{x\in\mathcal{X}} |x|^2 = 1$

Converse: $R > C$, $P_e \nrightarrow 0$

Converse part of Shannon's Noisy Coding Theorem

- We will show that rates $R > C$ are not achievable, i.e., if $R > C$ then P_e does not tend to zero as $n \to \infty$.
- In other words, in order to have $P_e \to 0$ we must have that $R \leq C$.
- The proof is based on Fano's inequality and the Data Processing Inequality.
- *Fano's inequality* relates probability of error and equivocation. We will derive it in two different manners over the next few slides.

Fano's Inequality



Error Probability and Equivocation

- Suppose we guess X from the observation Y. Let the guess be $\hat{X} = g(Y)$.
- **Conditional probability of error (conditioned on $Y = y$):**

$$P_e(y) = \Pr(X \neq \hat{X} | Y = y) = \sum_{\substack{x \in \mathcal{X}: \\ x \neq g(y)}} P_{X|Y}(x|y)$$

Probability of error:

$$P_e = P(X \neq \hat{X}) = \sum_{y \in \mathcal{Y}} P_Y(y) P_e(y)$$

- **Equivocation:**

$$H(X|Y) = \sum_{y \in \mathcal{Y}} P_Y(y) H(X|Y = y)$$

Fano's Inequality

Maximising $H(X|Y = y)$ for a given $P_e(y)$

$$\begin{aligned} P_e(y) &= \sum_{\substack{x\in\mathcal{X}:\\ x\neq g(y)}} P_{X|Y}(x|y) \\ &= 1 - P_{X|Y}(g(y)|y) \\ &\geq 1 - \max_{x\in\mathcal{X}} P_{X|Y}(x|y) \end{aligned}$$

Thus, for each $y \in \mathcal{Y}$, the probability of error $P_e(y)$ is minimised for

$$\hat{x} = g(y) = \arg\max_{x\in\mathcal{X}} P_{X|Y}(x|y),$$

for which $\max_{x\in\mathcal{X}} P_{X|Y}(x|y) = 1 - P_e$.

- Maximising $H(X|Y = y)$ for a given $P_e(y)$ is equivalent to maximising over all distributions for which $\max_{x\in\mathcal{X}} P_{X|Y}(x|y) = 1 - P_e$
- Entropy is maximised by the distribution that is uniform over the remaining symbols in the alphabet $\mathcal{X}$.

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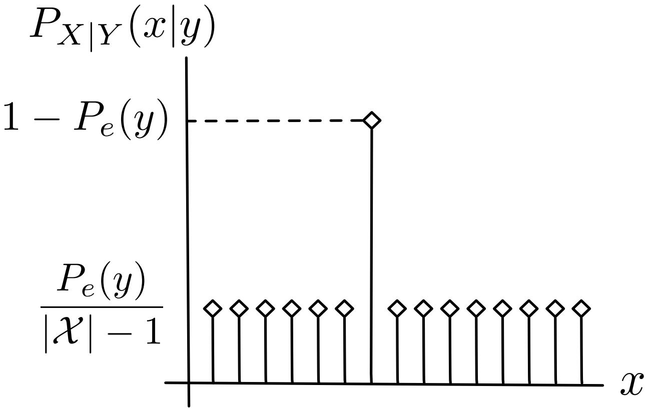


Fano's Inequality

Upper Bound (conditioned on $Y = y$)

$$H(X|Y=y) \le -(1-P_e(y))\log_2(1-P_e(y)) - (|\mathcal{X}|-1)\frac{P_e(y)}{|\mathcal{X}|-1}\log_2\frac{P_e(y)}{|\mathcal{X}|-1}$$
$$= H_b(P_e(y)) + P_e \log_2(|\mathcal{X}|-1) \tag{1}$$

Upper Bound (averaged over Y)

$$\begin{aligned}
H(X|Y) &= \sum_{y\in\mathcal{Y}} P_Y(y) H(X|Y=y) \\
&\le \sum_{y\in\mathcal{Y}} P_Y(y)\left[H_b(P_e(y)) + P_e(y)\log_2(|\mathcal{X}|-1)\right] \qquad \text{using (??)} \\
&\le H_b\left(\sum_{y\in\mathcal{Y}} P_Y(y)P_e(y)\right) + \log_2(|\mathcal{X}|-1)\sum_{y\in\mathcal{Y}} P_Y(y)P_e(y) \\
&= H_b(P_e) + P_e\log_2(|\mathcal{X}|-1)
\end{aligned}$$

where the third step follows by the concavity of $H_b(\cdot)$ an by Jensen's inequality.

$\Longrightarrow$ This can be weakened to $H(X|Y) \le 1 + P_e \log_2|\mathcal{X}|$.

Fano's Inequality

Standard proof of Fano's Inequality

The following alternative proof is much simpler but gives less insight than the one stated previously:

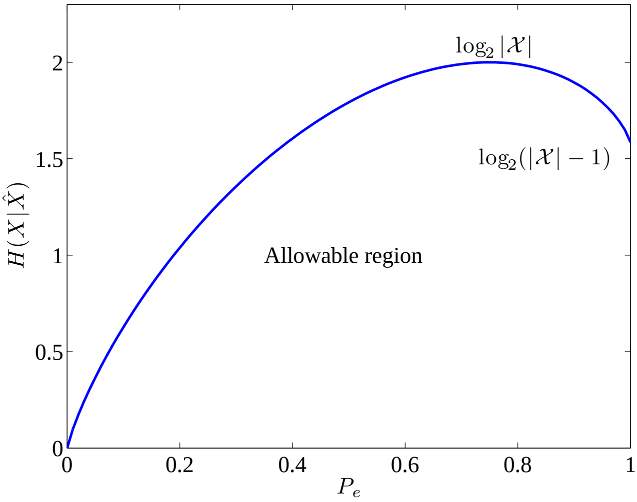
- Let E be an indicator random variable whose value is 0 if $\hat{X} = X$ and 1 if $\hat{X} \neq X$. Note that $P_E(1) = P_e$ and $H(E) = H_b(P_e)$.
- We have

$$
\begin{aligned}
H(X|Y) &= H(X,E|Y) - H(E|X,Y) && \text{chain rule for entropies} \\
&= H(X,E|Y) && \text{because } Y \text{ and } X \text{ determine } E \\
&= H(E|Y) + H(X|Y,E) && \text{chain rule for entropies} \\
&\leq H(E) + P_E(0)H(X|Y,E=0) + P_E(1)H(X|Y,E=1) \\
&\leq H_b(P_e) + P_e \log_2(|\mathcal{X}| - 1)
\end{aligned}
$$

  Here the fourth step follows because conditioning reduces entropy, and the last step follows because

  1. given $E = 0$, $g(Y) = \hat{X}$ determines X, which implies that $H(X|Y,E=0) = 0$
  2. given Y and $E = 1$, X can take on at most $|\mathcal{X}| - 1$ values. Hence, its entropy can be at most $\log_2(|\mathcal{X}| - 1)$.

Fano's Inequality



Converse Proof: $R > C$, $P_e \nrightarrow 0$

Proof of the converse part of Shannon's Noisy Coding Theorem

- Consider the above communications system. We have the Markov chain $\mathrm{m} \to \boldsymbol{X} \to \boldsymbol{Y} \to \hat{\mathrm{m}}$, where $\boldsymbol{X} = \phi(\mathrm{m})$ and $\hat{\mathrm{m}} = \varphi(\boldsymbol{Y})$.
- If m is drawn uniformly from the message set $\mathcal{M}$, then we have

$$\begin{aligned}
nR &= H(\mathrm{m}) && H(\mathrm{m}) = \log|\mathcal{M}| = nR \\
&= H(\mathrm{m}|\boldsymbol{Y}) + I(\mathrm{m};\boldsymbol{Y}) && I(\mathrm{m};\boldsymbol{Y}) = H(\mathrm{m}) - H(\mathrm{m}|\boldsymbol{Y}) \\
&\le 1 + P_e nR + I(\mathrm{m};\boldsymbol{Y}) && \text{by Fano's inequality} \\
&\le 1 + P_e nR + I(\boldsymbol{X};\boldsymbol{Y}) && \text{by Data Processing Inequality} \\
&\le 1 + P_e nR + \sum_{i=1}^{n} I(X_i;Y_i) && \text{because channel is memoryless} \\
&\le 1 + P_e nR + nC
\end{aligned}$$

where $C = \max_{P_X(\cdot)} I(X;Y)$.

- Rewriting the above equation yields

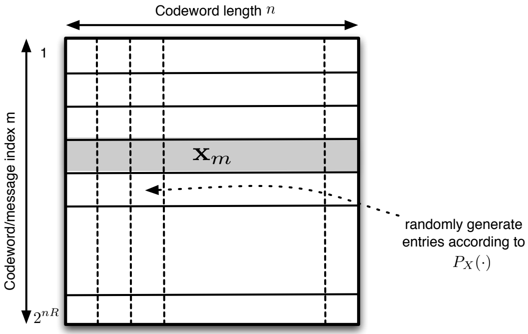
$$P_e \ge 1 - \frac{C}{R} - \frac{1}{nR}$$

Thus, if $R > C$, then P_e does not tend to zero as $n \to \infty$.

Achievability Proof $R < C, P_e \to 0$

Achievability Proof

- We show that rates $R < C$ are achievable, i.e., if $R < C$ then $P_e \to 0$ as $n \to \infty$.
- Codebook construction: generate codewords at random from a particular distribution. We will consider the case where the entries of each of the $|\mathcal{M}|$ codewords have been generated i.i.d. from $P_X(\cdot)$, i.e., $P_{\boldsymbol{X}}(\boldsymbol{x}) = \prod_{i=1}^{n} P_X(x_i)$.



Achievability Proof $R < C, P_e \to 0$

Achievability Proof

- We study maximum likelihood decoding, i.e., the decoder choses the message that maximises the likelihood of having been transmitted

$$\hat{\mathrm{m}} = \arg\max_{\mathcal{C}} P_{\boldsymbol{Y}|\boldsymbol{X}}(\boldsymbol{y}|\boldsymbol{x}_{\mathrm{m}})$$

- We study the average error probability over the ensemble of random codes, denoted by $\bar{P}_e = \frac{1}{|\mathcal{M}|}\sum_{\mathrm{m}=1}^{|\mathcal{M}|} \bar{P}_e(\mathrm{m})$
- Given the symmetry (random codewords), $\bar{P}_e = \bar{P}_e(\mathrm{m})$ for any $\mathrm{m} \in \mathcal{M}$
- Averaged over the random code ensemble, we have that

$$\begin{aligned}\bar{P}_e(\mathrm{m}) &= \mathbb{E}[\Pr\{\varphi(\boldsymbol{Y}) \neq \mathrm{m}|\boldsymbol{X}_{\mathrm{m}}, \boldsymbol{Y}\}] \\ &= \sum_{\boldsymbol{x}_{\mathrm{m}}}\sum_{\boldsymbol{y}} P_{\boldsymbol{X}}(\boldsymbol{x}_{\mathrm{m}})P_{\boldsymbol{Y}|\boldsymbol{X}}(\boldsymbol{y}|\boldsymbol{x}_{\mathrm{m}})\Pr\{\varphi(\boldsymbol{y}) \neq \mathrm{m}|\boldsymbol{x}_{\mathrm{m}}, \boldsymbol{y}\}\end{aligned}$$

where $\Pr\{\varphi(\boldsymbol{y}) \neq \mathrm{m}|\boldsymbol{x}_{\mathrm{m}}, \boldsymbol{y}\}$ is the probability that, for a channel output $\boldsymbol{y}$, the decoder φ selects a codeword other than the transmitted $\boldsymbol{x}_{\mathrm{m}}$

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Achievability Proof

- Using the union bound over all possible codewords, and for all $0 \leq \rho \leq 1$,

$$\Pr\{\varphi(\boldsymbol{y}) \neq \mathrm{m}|\boldsymbol{x}_\mathrm{m}, \boldsymbol{y}\} \leq \Pr\left\{\bigcup_{\mathrm{m}' \neq \mathrm{m}} \{\varphi(\boldsymbol{y}) = \mathrm{m}'|\boldsymbol{x}_\mathrm{m}, \boldsymbol{y}\}\right\}$$
$$\leq \left(\sum_{\mathrm{m}' \neq \mathrm{m}} \Pr\{\varphi(\boldsymbol{y}) = \mathrm{m}'|\boldsymbol{x}_\mathrm{m}, \boldsymbol{y}\}\right)^{\rho}$$

- The pairwise error probability $\Pr\{\varphi(\boldsymbol{y}) = \mathrm{m}'|\boldsymbol{x}_\mathrm{m}, \boldsymbol{y}\}$ of wrongly selecting message m' when message m has been transmitted and sequence $\boldsymbol{y}$ has been received is

$$\Pr\{\varphi(\boldsymbol{y}) = \mathrm{m}'|\boldsymbol{x}_\mathrm{m}, \boldsymbol{y}\} = \sum_{\boldsymbol{x}_{\mathrm{m}'} : P_{\boldsymbol{Y}|\boldsymbol{X}}(\boldsymbol{y}|\boldsymbol{x}_{\mathrm{m}'}) \geq P_{\boldsymbol{Y}|\boldsymbol{X}}(\boldsymbol{y}|\boldsymbol{x}_\mathrm{m})} P_{\boldsymbol{X}}(\boldsymbol{x}_{\mathrm{m}'})$$

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Achievability Proof $R < C, P_e \to 0$

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- As m′ is a dummy variable, for any $s > 0$ and $0 \leq \rho \leq 1$ it holds that

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- Hence, since $|\mathcal{M}| = 2^{nR}$ for any input distribution $P_X(x)$, and $0 \leq \rho \leq 1$,

$$\bar{P}_e \leq 2^{-n(E_0(\rho)-\rho R)} \quad \text{with} \quad E_0(\rho) \triangleq -\log_2 \mathbb{E}\left[\left(\sum_{x'} P_X(x')\left(\frac{P_{Y|X}(Y|x')}{P_{Y|X}(Y|X)}\right)^{\frac{1}{1+\rho}}\right)^{\rho}\right]$$

- $E_0(\rho)$ is called the Gallager function. The expectation is carried out according to the joint distribution $P_{X,Y}(x,y) = P_{Y|X}(y|x)P_X(x)$.
- We define the *random coding exponent* as

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- Hence, the tightest error probability bound is obtained as

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Achievability Proof $R < C, P_e \to 0$

Achievability Proof

- Using that $E_0(0) = 0$ we see that

$$\left.\frac{dE_0(\rho)}{d\rho}\right|_{\rho=0} = \lim_{\rho\to 0}\frac{E_0(\rho)}{\rho} = -\mathbb{E}\left[\log_2 \sum_{x'} P_X(x')\frac{P_{Y|X}(Y|x')}{P_{Y|X}(Y|X)}\right]$$

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- Actually, $0 < \frac{dE_0(\rho)}{d\rho} \le I(X;Y)$ with equality iff $\rho = 0$. Also, $\frac{d^2E_0(\rho)}{d\rho^2} \le 0$ for $\rho \ge 0$
- Then $E_0(\rho)$ is an *increasing* function of $\rho \ge 0$ and its maximum slope is at $\rho = 0$, given by $I(X;Y)$
- It follows that the function $g(\rho) = E_0(\rho) - \rho R$ has a maximum in $[0,1]$ if

$$\frac{dE_0(\rho)}{d\rho} - R = 0$$

has a solution in $[0,1]$. Otherwise, the maximum is achieved by $\rho = 1$ for $R < I(X;Y)$ or $\rho = 0$ for $R \ge I(X;Y)$

Achievability Proof $R < C, P_e \to 0$

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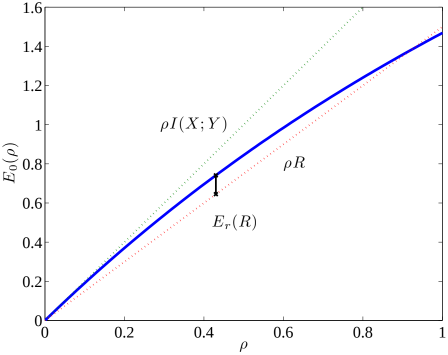
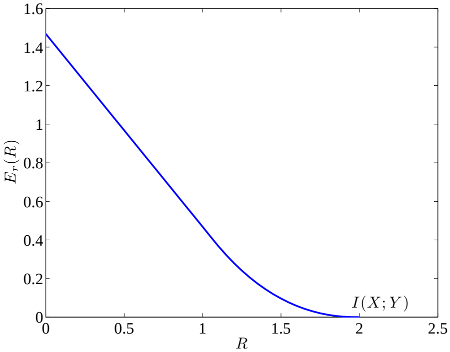
Achievability Proof $R < C$, $P_e \to 0$

- Since $\bar{P}_e \leq 2^{-nE_r(R)}$, then there must exist codes for which $P_e \leq 2^{-nE_r(R)}$
- Finally, since so far $P_X(X)$ was fixed, we can maximise over the input distribution to obtain the tightest bound and prove the achievability part of Shannon's theorem, i.e., rates $R < C$ are achievable.





Gallager function $E_0(\rho)$ and the random coding error exponent $E_r(R)$ for 16-QAM in an AWGN channel with SNR=5 dB

Channel Capacity Theorem

Summary

We have proved that the channel capacity is

$$C = \max_{p_X(X)} I(X;Y)$$

Achievability For every rate $R < C$ there exists a codebook $\mathcal{C}$ for which the probability of error tends to zero as $n \to \infty$.

Converse The probability of error satisfies

$$P_e \geq 1 - \frac{C}{R} - \frac{1}{nR}$$

Thus, if $R > C$ then the probability of error does not tend to zero as $n \to \infty$.

Summary The channel capacity is the fundamental limit of information transmission.