4F5: Advanced Wireless Communications

Handout 5: Low-Density Parity-Check Codes

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Lent 2012

Linear Block Codes

Definitions

- A binary code $\mathcal{C}$ of length n and dimension k is a set of different 2^k binary codewords of length n.
- The rate of the code is $R = \frac{1}{n}\log_2 |\mathcal{C}| = \frac{k}{n}$
- $\mathcal{C}$ is a vector subspace of the vector space defined by all possible binary vectors of length n, hence the code is linear
- $\mathcal{C}$ is the set of codewords $\boldsymbol{c}$ satisfying for all $\boldsymbol{b} \in \mathbb{F}_2^k$(row convention)

$$\boldsymbol{c} = \boldsymbol{b}\boldsymbol{G}, \quad \text{where} \quad \boldsymbol{G} = \begin{bmatrix} g_{1,1} & \cdots & g_{1,n} \\ g_{2,1} & \cdots & g_{2,n} \\ \vdots & \ddots & \vdots \\ g_{k,1} & \cdots & g_{k,n} \end{bmatrix} \text{ is the generator matrix}$$

- We can also express the code $\mathcal{C}$ as the set of codewords $\boldsymbol{c}$ such that

$$\boldsymbol{c}\boldsymbol{H}^T = \boldsymbol{0}, \quad \text{where} \quad \boldsymbol{H} = \begin{bmatrix} h_{1,1} & \cdots & h_{1,n} \\ h_{2,1} & \cdots & h_{2,n} \\ \vdots & \ddots & \vdots \\ h_{n-k,1} & \cdots & h_{n-k,n} \end{bmatrix} \text{ is the parity-check matrix}$$

- $\boldsymbol{H}$ represents the linear system of equations that every codeword must satisfy

Erasure Decoding Revisited

Decoding via the Parity-Check Matrix

$$\boldsymbol{H} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{y} = \begin{bmatrix} 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 1 & 0 & 0 & 0 & 1 & \varepsilon & 1 & \varepsilon & \varepsilon & \varepsilon & 1 \end{bmatrix}$$

Erasure Decoding Revisited

Decoding via the Parity-Check Matrix

$$
\boldsymbol{H} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
\boldsymbol{y} = \begin{bmatrix} 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 1 & 0 & 0 & 0 & 1 & \varepsilon & 1 & \varepsilon & \varepsilon & \varepsilon & 1 \end{bmatrix}
$$

$\longrightarrow$ **column permutation (2,3,4,5,11,13,14,15,1,6,7,8,9,10,12,16)**

Erasure Decoding Revisited

Permuted Columns (2,3,4,5,11,13,14,15,1,6,7,8,9,10,12,16)

$$\boldsymbol{H}' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{y}' = \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Erasure Decoding Revisited

Permuted Columns (2,3,4,5,11,13,14,15,1,6,7,8,9,10,12,16)

$$
\boldsymbol{H}' = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
\boldsymbol{y}' = \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}
$$

$\longrightarrow$ **Gauss elimination to upper triangular form**

Erasure Decoding Revisited

Permuted Columns (2,3,4,5,11,13,14,15,1,6,7,8,9,10,12,16)

$$
\boldsymbol{H}'' = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1
\end{bmatrix}
$$

$$
\boldsymbol{y}' = \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}
$$

Erasure Decoding Revisited

Permuted Columns (2,3,4,5,11,13,14,15,1,6,7,8,9,10,12,16)

$$
\boldsymbol{H}'' = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1
\end{bmatrix}
$$

$$
\boldsymbol{y}' = \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}
$$

Erasure Decoding Revisited

Permuted Columns (2,3,4,5,11,13,14,15,1,6,7,8,9,10,12,16)

$$
\boldsymbol{H}'' = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1
\end{bmatrix}
$$

$$
\boldsymbol{y}' = \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}
$$

Erasure Decoding Revisited

Permuted Columns (2,3,4,5,11,13,14,15,1,6,7,8,9,10,12,16)

$$
\boldsymbol{H}'' = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1
\end{bmatrix}
$$

$$
\boldsymbol{y}' = \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}
$$

Erasure Decoding Revisited

Permuted Columns (2,3,4,5,11,13,14,15,1,6,7,8,9,10,12,16)

$$
\boldsymbol{H}'' = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1
\end{bmatrix}
$$

$$
\boldsymbol{y}' = \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}
$$

Erasure Decoding Revisited

Permuted Columns (2,3,4,5,11,13,14,15,1,6,7,8,9,10,12,16)

$$
\boldsymbol{H}'' = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1
\end{bmatrix}
$$

$$
\boldsymbol{y}' = \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}
$$

Erasure Decoding Revisited

Permuted Columns (2,3,4,5,11,13,14,15,1,6,7,8,9,10,12,16)

$$
\boldsymbol{H}'' = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1
\end{bmatrix}
$$

$$
\boldsymbol{y}' = \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}
$$

Erasure Decoding Revisited

Permuted Columns (2,3,4,5,11,13,14,15,1,6,7,8,9,10,12,16)

$$
\boldsymbol{H}'' = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1
\end{bmatrix}
$$

$$
\boldsymbol{y}' = \begin{bmatrix} \varepsilon & \varepsilon & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}
$$

Erasure Decoding Revisited

Permuted Columns (2,3,4,5,11,13,14,15,1,6,7,8,9,10,12,16)

$$
\boldsymbol{H}'' = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1
\end{bmatrix}
$$

$$
\boldsymbol{y}' = \begin{bmatrix} \varepsilon & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}
$$

Erasure Decoding Revisited

Permuted Columns (2,3,4,5,11,13,14,15,1,6,7,8,9,10,12,16)

$$H'' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$$\mathbf{x}' = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Erasure Decoding Revisited

Permuted Columns (2,3,4,5,11,13,14,15,1,6,7,8,9,10,12,16)

$$H'' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{x}' = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$\longrightarrow$ **Un-permute columns**

Erasure Decoding Revisited

Permuted Columns (2,3,4,5,11,13,14,15,1,6,7,8,9,10,12,16)

$$H'' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{x}' = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$\longrightarrow$ **Un-permute columns**

$$\boldsymbol{x} = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{y} = \begin{bmatrix} 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 1 & 0 & 0 & 0 & 1 & \varepsilon & 1 & \varepsilon & \varepsilon & \varepsilon & 1 \end{bmatrix}$$

Erasure Decoding Revisited

Start again:Decoding via the Parity-Check Matrix, alternative approach

$$\boldsymbol{H} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{y} = \begin{bmatrix} 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 1 & 0 & 0 & 0 & 1 & \varepsilon & 1 & \varepsilon & \varepsilon & \varepsilon & 1 \end{bmatrix}$$

Erasure Decoding Revisited

Start again:Decoding via the Parity-Check Matrix, alternative approach

$$
\boldsymbol{H} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
\boldsymbol{y} = \begin{bmatrix} 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 1 & 0 & 0 & 0 & 1 & \varepsilon & 1 & \varepsilon & \varepsilon & \varepsilon & 1 \end{bmatrix}
$$

Erasure Decoding Revisited

Start again:Decoding via the Parity-Check Matrix, alternative approach

$$
\boldsymbol{H} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
\boldsymbol{y} = \begin{bmatrix} 0 & \varepsilon & 1 & \varepsilon & \varepsilon & 1 & 0 & 0 & 0 & 1 & \varepsilon & 1 & \varepsilon & \varepsilon & \varepsilon & 1 \end{bmatrix}
$$

Erasure Decoding Revisited

Start again:Decoding via the Parity-Check Matrix, alternative approach

$$
\boldsymbol{H} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
\boldsymbol{y} = \begin{bmatrix} 0 & \varepsilon & 1 & \varepsilon & \varepsilon & 1 & 0 & 0 & 0 & 1 & \varepsilon & 1 & \varepsilon & \varepsilon & \varepsilon & 1 \end{bmatrix}
$$

Erasure Decoding Revisited

Start again:Decoding via the Parity-Check Matrix, alternative approach

$$
\boldsymbol{H} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
\boldsymbol{y} = \begin{bmatrix} 0 & \varepsilon & 1 & \varepsilon & \varepsilon & 1 & 0 & 0 & 0 & 1 & {\color{red}0} & 1 & \varepsilon & \varepsilon & \varepsilon & 1 \end{bmatrix}
$$

Erasure Decoding Revisited

Start again:Decoding via the Parity-Check Matrix, alternative approach

$$\boldsymbol{H} = \begin{bmatrix} 0&1&0&0&0&0&0&0&0&0&0&1&1&0&0&1 \\ 0&0&0&1&0&1&1&0&0&1&0&0&0&0&0&0 \\ 0&0&1&0&1&0&0&0&1&0&0&0&0&1&0&0 \\ 1&0&0&0&0&0&0&1&0&0&0&0&0&1&0&0 \\ 1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1 \\ 0&1&0&0&0&0&1&0&0&0&1&0&0&0&1&0 \\ 0&0&0&1&0&0&0&1&0&0&0&0&0&0&1&0 \\ 0&0&1&0&0&0&0&0&1&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&1&0&0&1&1&0&0 \\ 1&0&0&1&0&0&0&1&0&0&1&0&0&0&0&0 \end{bmatrix}$$

$$\boldsymbol{y} = \begin{bmatrix} 0&\varepsilon&1&\varepsilon&\varepsilon&1&0&0&0&1&0&1&\varepsilon&\varepsilon&\varepsilon&1 \end{bmatrix}$$

Erasure Decoding Revisited

Start again:Decoding via the Parity-Check Matrix, alternative approach

$$
\boldsymbol{H} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
\boldsymbol{y} = \begin{bmatrix} 0 & \varepsilon & 1 & 0 & \varepsilon & 1 & 0 & 0 & 0 & 1 & 0 & 1 & \varepsilon & \varepsilon & \varepsilon & 1 \end{bmatrix}
$$

Erasure Decoding Revisited

Start again:Decoding via the Parity-Check Matrix, alternative approach

$$
\boldsymbol{H} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
\boldsymbol{y} = \begin{bmatrix} 0 & \varepsilon & 1 & 0 & \varepsilon & 1 & 0 & 0 & 0 & 1 & 0 & 1 & \varepsilon & \varepsilon & \varepsilon & 1 \end{bmatrix}
$$

Erasure Decoding Revisited

Start again: Decoding via the Parity-Check Matrix, alternative approach

$$
\boldsymbol{H} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
\boldsymbol{y} = \begin{bmatrix} 0 & \varepsilon & 1 & 0 & \varepsilon & 1 & 0 & 0 & 0 & 1 & 0 & 1 & \varepsilon & \varepsilon & 0 & 1 \end{bmatrix}
$$

Erasure Decoding Revisited

Start again:Decoding via the Parity-Check Matrix, alternative approach

$$
\boldsymbol{H} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
\boldsymbol{y} = \begin{bmatrix} 0 & \varepsilon & 1 & 0 & \varepsilon & 1 & 0 & 0 & 0 & 1 & 0 & 1 & \varepsilon & \varepsilon & 0 & 1 \end{bmatrix}
$$

Erasure Decoding Revisited

Start again:Decoding via the Parity-Check Matrix, alternative approach

$$
\boldsymbol{H} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
\boldsymbol{y} = \begin{bmatrix} 0 & \varepsilon & 1 & 0 & \varepsilon & 1 & 0 & 0 & 0 & 1 & 0 & 1 & \varepsilon & 0 & 0 & 1 \end{bmatrix}
$$

Erasure Decoding Revisited

Start again:Decoding via the Parity-Check Matrix, alternative approach

$$
\boldsymbol{H} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
\boldsymbol{y} = \begin{bmatrix} 0 & \varepsilon & 1 & 0 & \varepsilon & 1 & 0 & 0 & 0 & 1 & 0 & 1 & \varepsilon & 0 & 0 & 1 \end{bmatrix}
$$

Erasure Decoding Revisited

Start again:Decoding via the Parity-Check Matrix, alternative approach

$$\boldsymbol{H} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

$$\boldsymbol{y} = \begin{bmatrix} 0 & \varepsilon & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & \varepsilon & 0 & 0 & 1 \end{bmatrix}$$

Erasure Decoding Revisited

Start again:Decoding via the Parity-Check Matrix, alternative approach

$$
\boldsymbol{H} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
\boldsymbol{y} = \begin{bmatrix} 0 & \varepsilon & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & \varepsilon & 0 & 0 & 1 \end{bmatrix}
$$

Erasure Decoding Revisited

Start again:Decoding via the Parity-Check Matrix, alternative approach

$$
\boldsymbol{H} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
\boldsymbol{y} = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & \varepsilon & 0 & 0 & 1 \end{bmatrix}
$$

Erasure Decoding Revisited

Start again:Decoding via the Parity-Check Matrix, alternative approach

$$
\boldsymbol{H} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
\boldsymbol{y} = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & \varepsilon & 0 & 0 & 1 \end{bmatrix}
$$

Erasure Decoding Revisited

Start again:Decoding via the Parity-Check Matrix, alternative approach

$$
\boldsymbol{H} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
\boldsymbol{x} = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}
$$

Erasure Decoding Revisited

Start again:Decoding via the Parity-Check Matrix, alternative approach

$$
H = \begin{bmatrix}
0&1&0&0&0&0&0&0&0&0&0&1&1&0&0&1\\
0&0&0&1&0&1&1&0&0&1&0&0&0&0&0&0\\
0&0&1&0&1&0&0&0&1&0&0&0&0&1&0&0\\
1&0&0&0&0&0&0&1&0&0&0&0&0&1&0&0\\
1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1\\
0&1&0&0&0&0&1&0&0&0&1&0&0&0&1&0\\
0&0&0&1&0&0&0&1&0&0&0&0&0&0&1&0\\
0&0&1&0&0&0&0&0&1&0&0&1&0&0&0&0\\
0&0&0&0&1&0&0&0&0&1&0&0&1&1&0&0\\
1&0&0&1&0&0&0&1&0&0&1&0&0&0&0&0
\end{bmatrix}
$$

$$
\boldsymbol{x} = \begin{bmatrix} 0&0&1&0&1&1&0&0&0&1&0&1&0&0&0&1 \end{bmatrix}
$$

$\longrightarrow$resolve parity-check equations with 1 erasure

Discussion

Comparison of Decoder 1 (triangulation) vs. Decoder 2 (resolve parity-check equations of $\boldsymbol{H}$ with 1 erasure)

- Both decoders operate by resolving parity-check equations with 1 erasure
- The second decoder uses only the $(N-K)$ parity-check equations in $\boldsymbol{H}$
- The first decoder constructs additional parity-check equations from among the $2^{N-K}-1$ non-trivial linear combinations of the parity-check equations of $\boldsymbol{H}$ specifically so that it can successively resolve all erasures
- The first decoder is an optimal ML decoder and is superior to the second decoder
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Comparison of Decoder 1 (triangulation) vs. Decoder 2 (resolve parity-check equations of $\boldsymbol{H}$ with 1 erasure)

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Decoding Tree

$$
\boldsymbol{H} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

Analysis of successful decoding

- X_2 can be resolved if Y_2 is not an erasure OR if Equations E_1 or E_6 can be resolved
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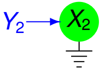
- etc.



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$$
\boldsymbol{H} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
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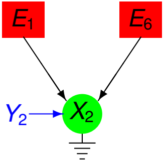

Decoding Tree

$$
\boldsymbol{H} = \begin{bmatrix}
0 & \color{red}{\mathbf{1}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \color{red}{\mathbf{1}} & \color{red}{\mathbf{1}} & 0 & 0 & \color{red}{\mathbf{1}} \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
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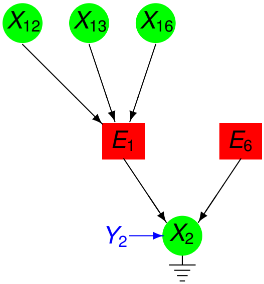

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0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
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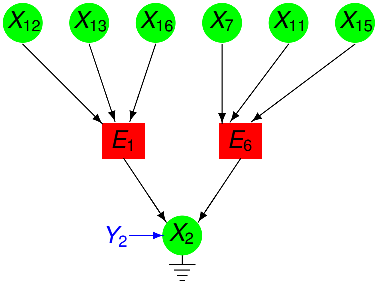

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0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
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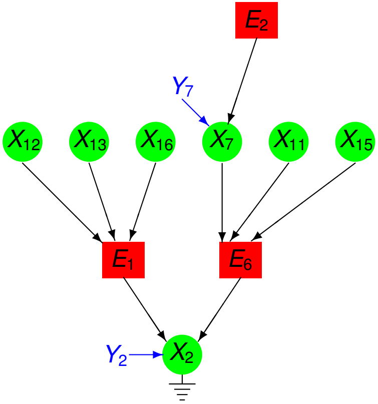

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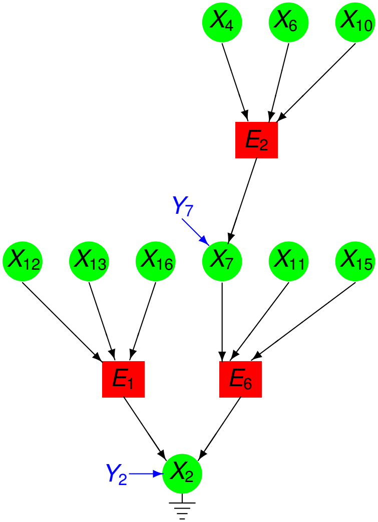
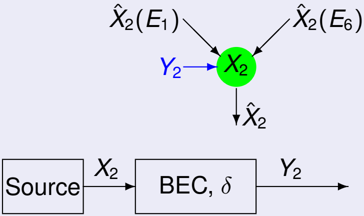
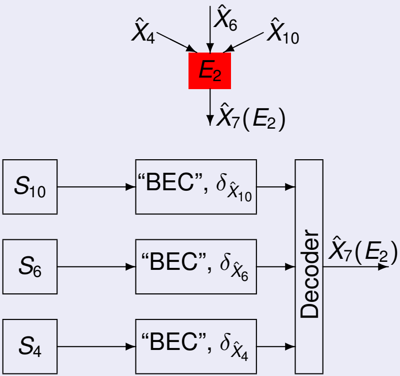


Tree Elements

Variable Node (or Bit Node) = Repetition Code



Constraint Node (or Check Node) = Single Parity-Check Code



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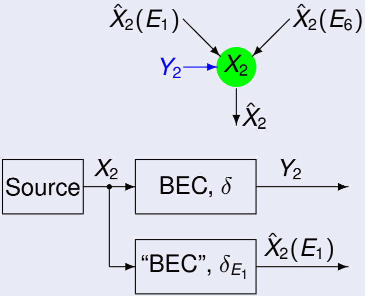

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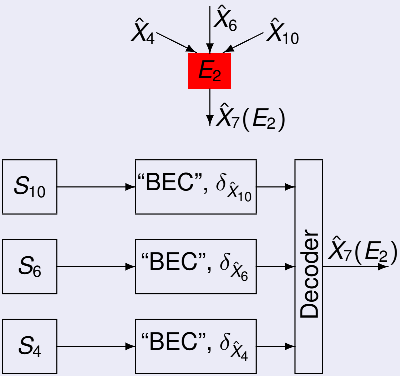

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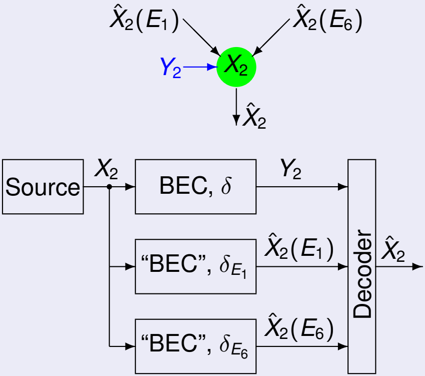

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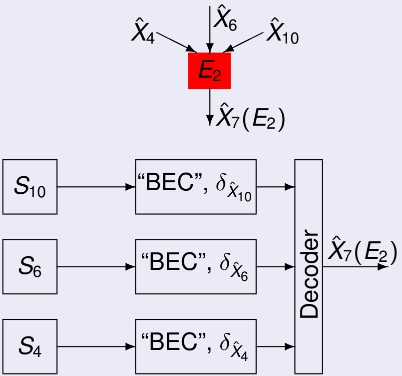

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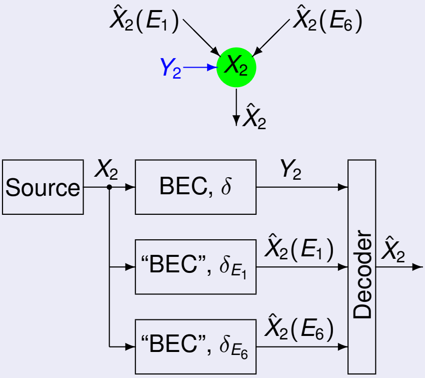

- The decoder can resolve X_2 if any of its inputs is not erased

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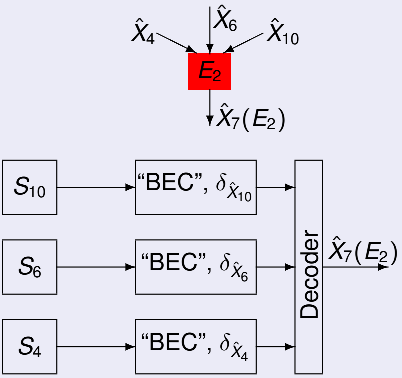

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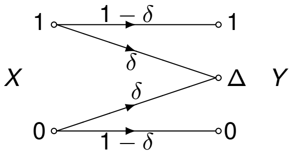
Tree Elements

Binary Erasure Channel (BEC)



$$\boldsymbol{H} = \begin{bmatrix} 0&1&0&0&0&0&0&0&0&0&0&1&1&0&0&1 \\ 0&0&0&1&0&1&1&0&0&1&0&0&0&0&0&0 \\ 0&0&1&0&1&0&0&0&1&0&0&0&0&1&0&0 \\ 1&0&0&0&0&0&0&1&0&0&0&0&0&1&0&0 \\ 1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1 \\ 0&1&0&0&0&0&1&0&0&0&1&0&0&0&1&0 \\ 0&0&0&1&0&0&0&1&0&0&0&0&0&0&1&0 \\ 0&0&1&0&0&0&0&0&1&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&1&0&0&1&1&0&0 \\ 1&0&0&1&0&0&0&1&0&0&1&0&0&0&0&0 \end{bmatrix}$$

Linear Block Codes
Definitions

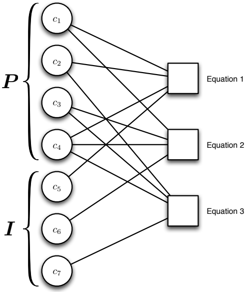
- Hamming $(7,4)$ code of rate $R = \frac{4}{7}$, parity-check matrix given by

$$\boldsymbol{H} = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

- The parity-check matrix implies that all codewords satisfy $\boldsymbol{cH}^T = 0$

$$c_1 + c_2 + c_4 + c_5 = 0$$
$$c_1 + c_3 + c_4 + c_6 = 0$$
$$c_2 + c_3 + c_4 + c_7 = 0$$

- Factor graph representation: bit nodes $\bigcirc$, check nodes $\square$, edges if there is a '1')



LDPC Basics

LDPC Basics

- A bit node has degree d if it is connected to d check nodes (influences d parity-check equations)
- A check node has degree d if it is connected to d bit nodes (the parity-check equation has d variables)
- Regular LDPC: all bit nodes have the same degree, all check nodes have the same degree
- Irregular LDPC: different bit (check) nodes can have different degrees
- The $(7,4)$ Hamming code has 3 bit nodes of degree 1, 3 bit nodes of degree 2 and 1 bit node of degree 3. All 3 check nodes have degree 4.
- Typically degree distributions are represented by polynomials
  - $\Lambda(x) = \sum_i^{d^\ell_{\max}} \Lambda_i x^i = 3x + 3x^2 + x^3, \quad P(x) = \sum_i^{d^r_{\max}} P_i x^i = 3x^4$ (total number of nodes)
  - $L(x) = \sum_i^{d^\ell_{\max}} L_i x^i = \frac{3}{7}x + \frac{3}{7}x^2 + \frac{1}{7}x^3, \quad R(x) = \sum_i^{d^r_{\max}} R_i x^i = \frac{3}{3}x^4$ (fraction of nodes)
  - $\lambda(x) = \sum_i^{d^\ell_{\max}} \lambda_i x^{i-1} = \frac{3}{12} + \frac{6}{12}x + \frac{3}{12}x^2, \ \rho(x) = \sum_i^{d^r_{\max}} \rho_i x^{i-1} = \frac{12}{12}x^3$ (edge fraction)

LDPC Basics

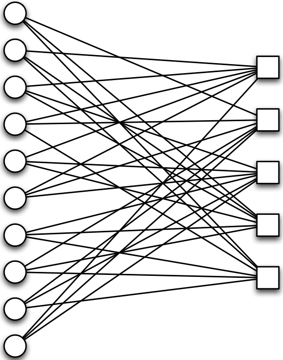
- Let N_e denote the number of edges in the graph (an edge corresponds to a '1' in the parity check matrix)
- Then, $\sum_i i\Lambda_i = \sum_i iP_i = N_e$, $\Lambda(1) = n$ and $P(1) = n(1-R)$
- Average degrees are $\bar{d}^\ell = L'(1) = \frac{1}{\int_0^1 \lambda(x)dx}$ and $\bar{d}^r = R'(1) = \frac{1}{\int_0^1 \rho(x)dx}$
- Hence $R = 1 - \dfrac{P(1)}{\Lambda(1)} = 1 - \dfrac{L'(1)}{R'(1)} = 1 - \dfrac{\bar{d}^\ell}{\bar{d}^r} = 1 - \dfrac{\int_0^1 \rho(x)dx}{\int_0^1 \lambda(x)dx} = 1 - \dfrac{3}{7} = \dfrac{4}{7}$
- Define LDPC(n, λ, ρ) as the ensemble of all possible LDPC codes of length n generated according to $\lambda(x), \rho(x)$, i.e., the probability that a randomly selected edge is connected to a bit (check) node of degree i is λ_i (ρ_i)
- Regular LDPC, $\lambda(x), \rho(x)$ monomials

LDPC Basics

Example

LDPC$(10, x^2, x^5)$ is also known as the $(3, 6)$-Gallager code of length 10. The rate is $R = 1 - \frac{3}{6} = \frac{1}{2}$ and a possible parity-check matrix is given below

$$\boldsymbol{H} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

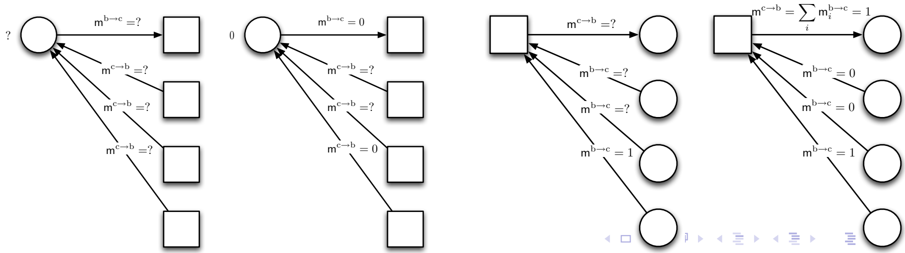


Iterative Decoding over the BEC

Message Passing Algorithm

1. Initialisation
   - Set all check-to-bit node messages $m^{c \to b} = ?$
2. Bit-to-check node messages (1/2 iteration)
   - For all erased bit nodes $m^{b \to c} = \begin{cases} ? & \text{if all incoming messages are '?'} \\ m^{c \to b} & \text{if one or more incoming messages} \neq ? \end{cases}$
3. Check-to-bit node messages (1/2 iteration)
   - For all check nodes $m^{c \to b} = \begin{cases} ? & \text{if one or more incoming messages are ?} \\ \sum_i m_i^{b \to c} & \text{if all incoming messages} \neq ? \end{cases}$

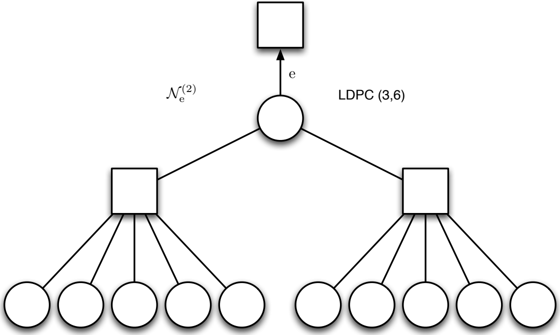
Note that outgoing messages do not depend on the incoming message to that node



Iterative Decoding over the BEC

Error Probability Analysis

- It is not difficult to see that the error probability of the iterative decoder does not depend on the transmitted codeword, hence we assume the all-zero codeword
- We will analyse how the error probability of the iterative decoder evolves through the iterations
- Directed neighbourhood of an edge $e = \mathrm{b} \rightarrow \mathrm{c}$, of depth d denoted by $\mathcal{N}_e^{(d)}$ is the subgraph including all paths of length d starting from bit node b and not including the edge e
- $\mathcal{N}_e^{(d)}$ is a tree if all nodes are distinct: no cycles, closed paths



Iterative Decoding over the BEC

Error Probability Analysis

- Assume $\mathcal{N}_e^{(d)}$ cycle free ($n \to \infty$ case). Then input messages independent.
- A message sent by bit node b along edge e at iteration ℓ is a function of the messages propagated through $\mathcal{N}_e^{(2\ell)}$
- The error probability at iteration ℓ is given by

$$P_b^{(\ell)} = \sum_{\boldsymbol{y}} \Pr\left\{ \mathrm{m}_e^{\mathrm{b}\to\mathrm{c}} = ? \mid \mathcal{N}_e^{(2\ell)} \text{cycle free}, \mathrm{LDPC}(\lambda, \rho), \boldsymbol{y} \right\} \Pr\{\mathrm{LDPC}(\lambda,\rho)\} P_{\boldsymbol{Y}}(\boldsymbol{y})$$

- Assume $P_b^{(\ell-1)}$ is known and $P_b^{(0)} = p$
  - $\mathrm{m}^{\mathrm{c}\to\mathrm{b}} = ?$ for a degree-j check node if some of the $j-1$ incoming $\mathrm{m}^{\mathrm{b}\to\mathrm{c}} = ?$

$$q_j = \Pr\{\mathrm{m}^{\mathrm{c}\to\mathrm{b}} = ? | \text{check node of degree } j\} = 1 - \left(1 - P_b^{(\ell-1)}\right)^{j-1}$$

  - Averaging over all possible check node degrees

$$q = \sum_j \rho_j q_j = 1 - \rho\left(1 - P_b^{(\ell-1)}\right)$$

  - $\mathrm{m}^{\mathrm{b}\to\mathrm{c}} = ?$ for a degree-i bit node if all $i-1$ incoming messages are erasures

$$p_i = \Pr\{\mathrm{m}^{\mathrm{b}\to\mathrm{c}} = ? | \text{bit node of degree } i\} = p\, q^{i-1}$$

  - Averaging over all possible bit node degrees

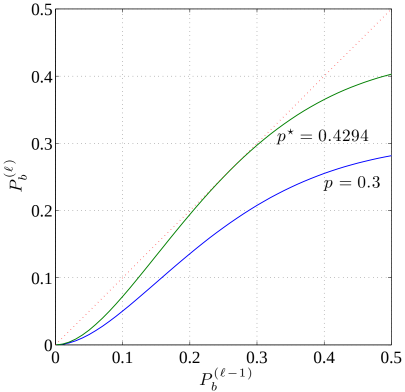
$$P_b^{(\ell)} = \sum_i \lambda_i p_i = p\lambda\left(1 - \rho\left(1 - P_b^{(\ell-1)}\right)\right)$$

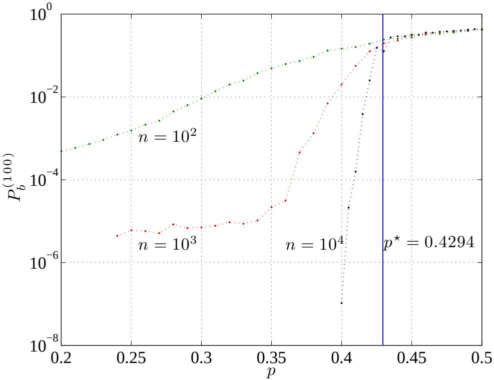
  - Remember $\lambda(x) = \sum_i \lambda_i x^{i-1}$, $\rho(x) = \sum_j \rho_j x^{j-1}$

Iterative Decoding over the BEC

Error Probability Analysis

- The figure below (left) shows the recursion $P_b^{(\ell)}$ vs. $P_b^{(\ell-1)}$ for LDPC(3,6). $P_b^{(0)} = p$
- Threshold: $p^\star = \sup\left\{p \in [0,1] \ : \ \lim\limits_{\ell\to\infty} P_b^{(\ell)} = 0\right\}$
  - for $p < p^\star$ then $\lim_{\ell\to\infty} P_b^{(\ell)} = 0$ (single fixed point at zero)
  - for $p > p^\star$ then $\lim_{\ell\to\infty} P_b^{(\ell)} > 0$ (multiple fixed point)
- LDPC codes (degree distributions) can be optimised such that the P_b recursion is closest to $y = x$





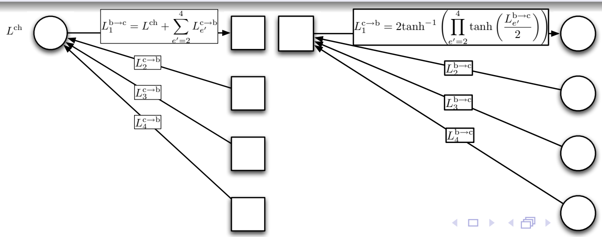
Message Passing and General Channels

Message Passing

- The iterative decoder for the BEC is an example of *message passing* decoders
- For other channels like AWGN, the decoder is different from the BEC one
- Since variables are binary, we can consider log-likelihood ratios

$$L_i^{\text{ch}} = \log \frac{P_{Y|X}(y_i| + 1)}{P_{Y|X}(y_i| - 1)}$$

1. Bit-to-check node messages (1/2 iteration)
$$L_e^{\text{b}\to\text{c}} = L_i^{\text{ch}} + \sum_{e' \neq e} L_{e'}^{\text{c}\to\text{b}}$$
2. Check-to-bit node messages (1/2 iteration)
$$L_e^{\text{c}\to\text{b}} = 2\tanh^{-1}\left(\prod_{e' \neq e} \tanh\left(\frac{L_{e'}^{\text{b}\to\text{c}}}{2}\right)\right)$$
3. Final decision $\hat{x}_i = \text{sign}\left(L_i^{\text{ch}} + \sum_e L_e^{\text{c}\to\text{b}}\right)$



Message Passing and General Channels