4F5: Advanced Wireless Communications

Handout 8: Transmission over Fading Channels

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Lent 2012

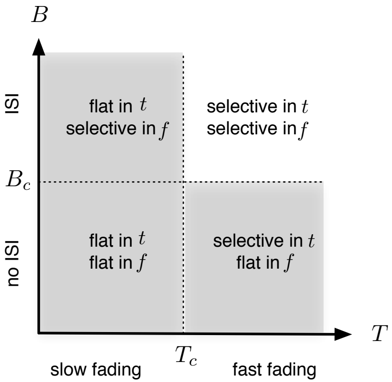
Underspread Channels

Considered channel models

We consider

1. frequency-flat fading
2. slow frequency-selective fading

Most common scenarios for underspread channels.



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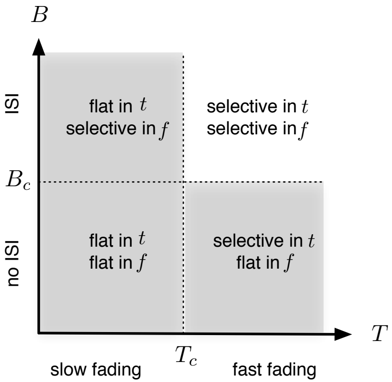

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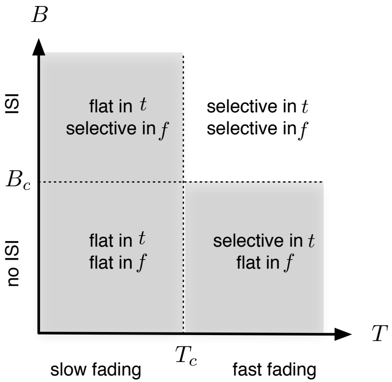


Frequency-Flat Fading

Channel Model

Channel output is given by

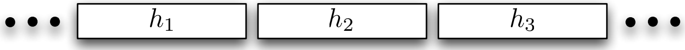
$$y_i = h_i x_i + n_i, \qquad i = 1, 2, \dots$$

- Slow fading: $h_i = h$.
- Fast fading: $h_1, h_2, \dots$ are i.i.d.
- Block fading: h_i is constant for T_c time slots and changes then independently, i.e.,

$$y_i = h_{\lceil i/T_c \rceil} x_i + n_i, \qquad i = 1, 2, \dots$$

  where $h_1, h_2, \dots$ are i.i.d.
- Receiver has perfect knowledge of $h_1, h_2, \dots$: the channel can be estimated by transmitting pilot symbols.
- Transmitter has no knowledge of $h_1, h_2, \dots$: receiver would need to transfer channel estimates to transmitter.

Block fading channel



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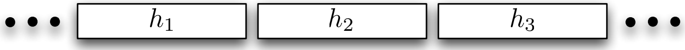

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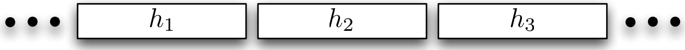

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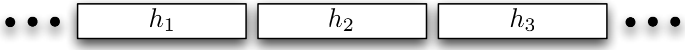

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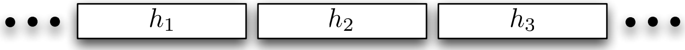

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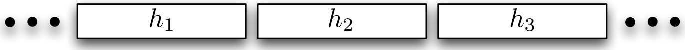

Performance Analysis

- Assume that $h_1, h_2, \ldots$ are Gaussian with mean zero and variance 1. This is referred to as Rayleigh fading.
- Consider uncoded BPSK modulation over a Rayleigh fading channel. (Whether the fading is fast or slow does not matter. Why?)
- Let $P_b(h)$ denote the probability of error when $h_i = h$. The average probability of error (averaged over h_i) is given by

$$P_b = \int f_H(h) P_b(h) \mathrm{d}h$$

- If $|h_i|^2 = 1$ with probability one, then the channel specialises to the AWGN channel for which $P_b^{(G)} = Q\left(\sqrt{2\mathrm{SNR}}\right)$.
- If $|h_i| > 1$ then $P_b(h_i) < P_b^{(G)}$, and if $|h_i| < 1$ then $P_b(h_i) > P_b^{(G)}$. By the convexity of $P_b(h)$ it follows that having a nondeterministic channel gain is detrimental.
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$$P_b = \int f_H(h) Q\left(\sqrt{2|h|^2\mathrm{SNR}}\right) \mathrm{d}h = \frac{1}{2}\left(1 - \sqrt{\frac{\mathrm{SNR}}{1+\mathrm{SNR}}}\right) \approx \frac{1}{4\mathrm{SNR}}$$

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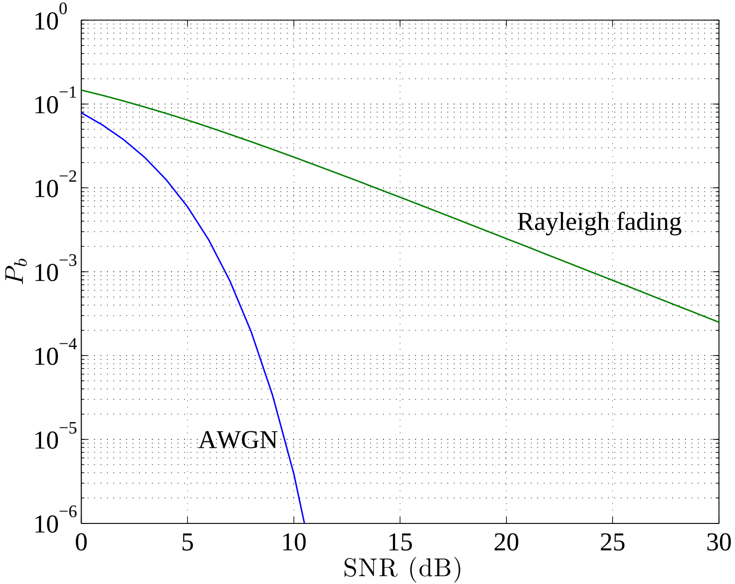
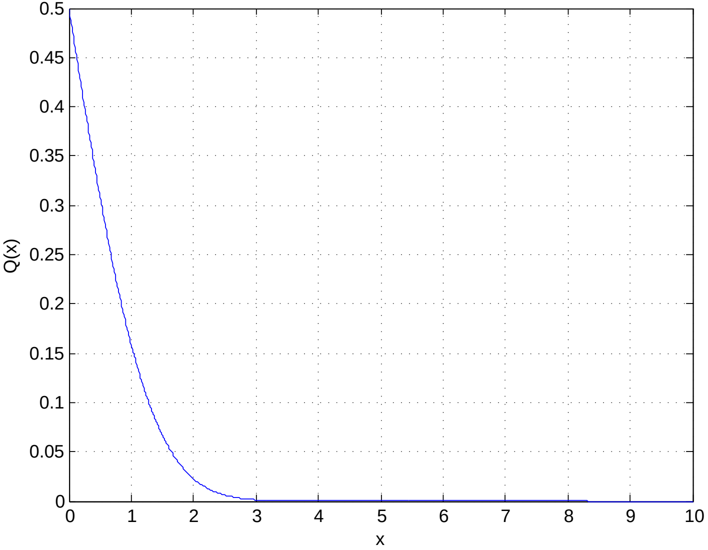
Probability of Error



Uncoded Transmission

Q-Function is Convex



Diversity

Repetition Code

- Bad error performance is due to events where h_i is small.
- Suppose we transmit the same BPSK symbol twice: we thus observe

$$y_1 = h_1 x + n_1 \qquad \text{and} \qquad y_2 = h_2 x + n_2$$

- If h_1 and h_2 are independent, then it is less likely that both h_1 and h_2 are small.
- Maximum-ratio combining:

$$h_1^* y_1 + h_2^* y_2 = \left(|h_1|^2 + |h_2|^2\right) x + h_1^* n_1 + h_2^* n_2$$

  maximizes the signal-to-noise ratio at the receiver.
- The probability of error is upper-bounded by

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Diversity Techniques

- If the receiver observes D independently faded replicas of the same information signal, the probability of all deep fades is reduced significantly: $P_b \propto \text{SNR}^{-D}$
- Diversity order $d \triangleq \lim_{\text{SNR}\to\infty} -\frac{\log P_b}{\log \text{SNR}}$ (asymptotic slope of P_e in a log-log scale)
- There are multiple ways of achieving diversity in practical wireless systems:

  Frequency: transmitting the same information over D carriers
  Time: transmitting the same information over D time slots
  Space: equip the receiver, transmitter or both with multiple antennas

- **Note:** time diversity does not require repetition coding. We will see later that usual channel coding works well.
- We have to ensure that we observe independently faded replicas of the transmitted signal. To this end, the carriers/time slots/antennas must be spaced sufficiently apart from each other.

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Diversity Techniques

- If the receiver observes D independently faded replicas of the same information signal, the probability of all deep fades is reduced significantly: $P_b \propto \text{SNR}^{-D}$
- Diversity order $d \triangleq \lim_{\text{SNR}\to\infty} -\frac{\log P_b}{\log \text{SNR}}$ (asymptotic slope of P_e in a log-log scale)
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Multiple Receive Antennas

- Suppose we have 1 transmit and N_R receive antennas. Then,

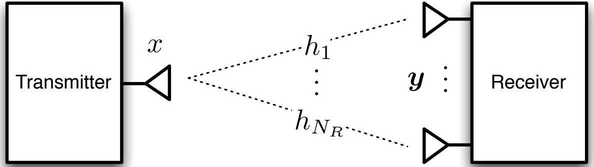
$$\boldsymbol{y}_i = \boldsymbol{h}_i x_i + \boldsymbol{n}_i, \qquad i = 1, 2, \dots$$

- With maximum-ratio combining we obtain

$$\boldsymbol{h}_i^H \boldsymbol{y}_i = \|\boldsymbol{h}_i\|^2 x_i + \boldsymbol{h}_i^H \boldsymbol{n}_i = \left(|h_i(1)|^2 + \ldots + |h_i(N_R)|^2\right) x_i + \widetilde{n}_i$$

- The error probability is upper-bounded by

$$P_b \le \frac{1}{2\left(1 + \frac{\text{SNR}}{N_R}\right)^{N_R}} \propto \text{SNR}^{-N_R}$$



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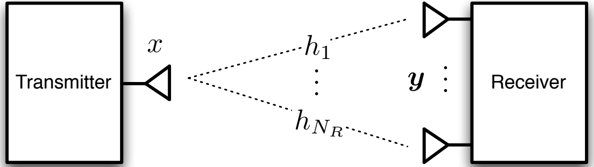

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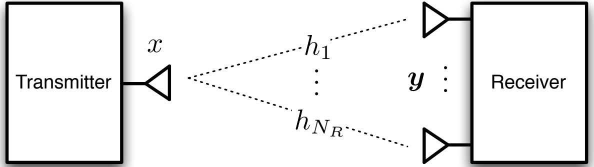

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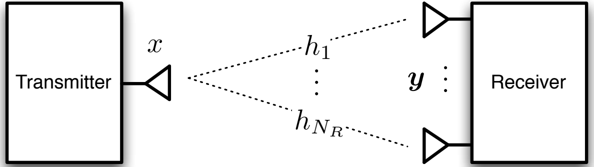
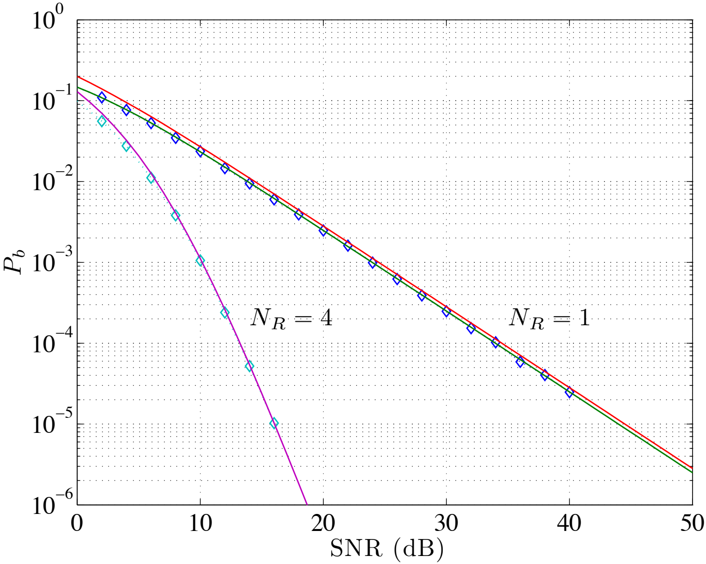


Diversity

Probability of Error: Multiple Receive Antennas



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- Suppose we have N_T transmit and 1 receive antennas, and assume that the fading is slow. Then,

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Alamouti's Scheme

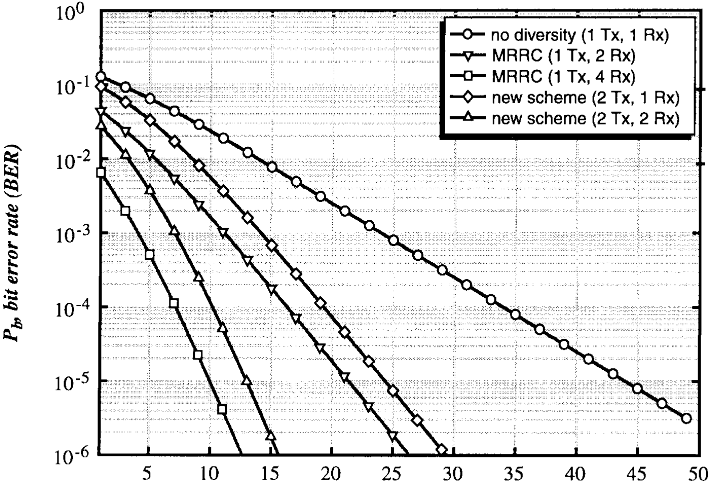

Image from S. M. Alamouti, "A simple transmit diversity technique for wireless communications," *IEEE Journal on Select. Areas in Communications*, October 1998.

Multiple-Input Multiple-Output (MIMO)

- Suppose we have N_T transmit and N_R receive antennas.
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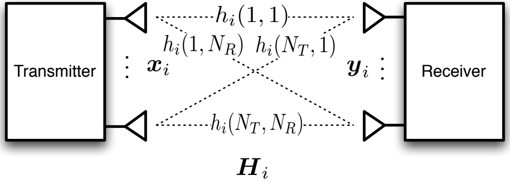

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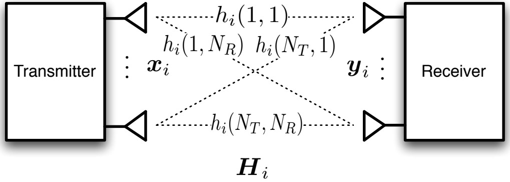

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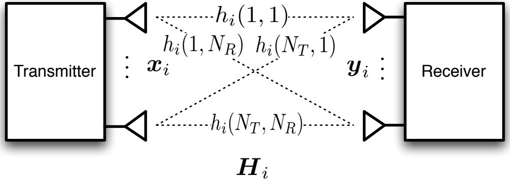

Diversity

Multiple-Input Multiple-Output (MIMO)

- Suppose we have N_T transmit and N_R receive antennas.
- The output of the multiple-input multiple-output (MIMO) flat fading channel is

$$\boldsymbol{y}_i = \boldsymbol{H}_i \boldsymbol{x}_i + \boldsymbol{n}_i, \qquad i = 1, 2, \ldots$$

where

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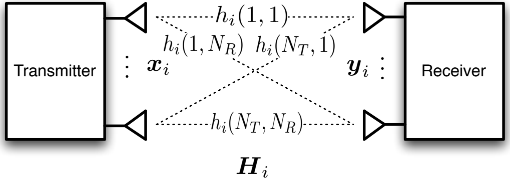


Frequency-Selective Fading

Channel Model

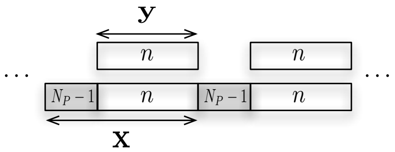
- For slow frequency-selective fading, the channel output is

$$y_i = \sum_{\ell=0}^{N_P-1} h_\ell x_{i-\ell} + n_i, \qquad i = 1, 2, \ldots$$

- Thus, each channel realisation is a random realisation of an ISI channel with N_P taps having zero mean and variance given by the multipath delay profile $R_h(\tau)$.
- Since the fading is slow, the taps do not depend on i.

Orthogonal Frequency-Division Multiplexing (OFDM)

- Converts ISI channel into parallel frequency-flat fading channels.
- Transmit block-wise in blocks of n symbols.
- Before each block introduce a guard period of $N_P - 1$.



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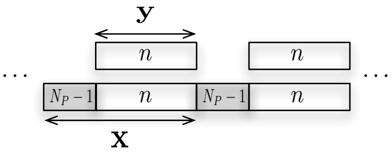

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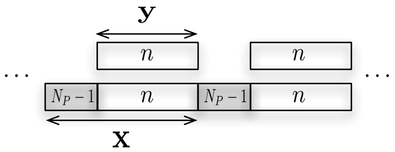

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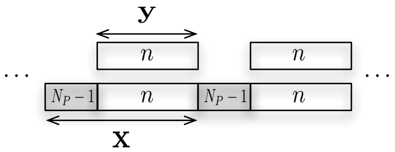

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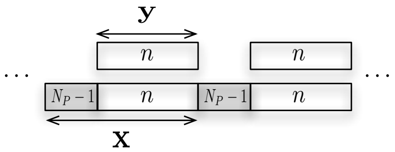

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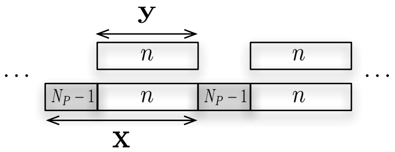

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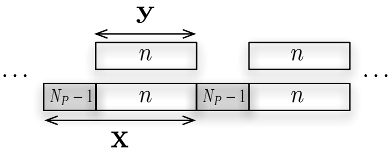

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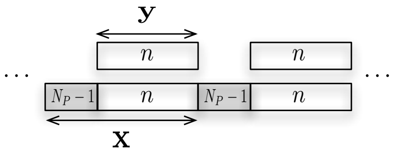

Frequency-Selective Fading

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- Let $\boldsymbol{y} = (y_1, \ldots, y_n)^T$, $\boldsymbol{n} = (n_1, \ldots, n_n)^T$ and $\boldsymbol{x} = \left(\boldsymbol{x}^{(p)}, \boldsymbol{x}^{(i)}\right)^T$.
- In matrix notation

$$\boldsymbol{y} = \begin{pmatrix} h_{N_P-1} & \cdots & h_0 & 0 & \cdots & \cdots & 0 \\ 0 & h_{N_P-1} & \cdots & h_0 & 0 & \cdots & 0 \\ \vdots & \ddots & & \ddots & \ddots & & \vdots \\ 0 & \cdots & \cdots & 0 & h_{N_P-1} & \cdots & h_0 \end{pmatrix} \boldsymbol{x} + \boldsymbol{n}$$

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$$\begin{aligned} \boldsymbol{F}_n\boldsymbol{y} &= \boldsymbol{F}_n\left(\widetilde{\boldsymbol{H}}\boldsymbol{x}^{(i)} + \boldsymbol{n}\right) \\ &= \boldsymbol{F}_n\boldsymbol{F}_n^H \operatorname{diag}(H_0,\dots,H_{n-1})\, \boldsymbol{F}_n\boldsymbol{x}^{(i)} + \boldsymbol{F}_n\boldsymbol{n} \\ &= \operatorname{diag}(H_0,\dots,H_{n-1})\widetilde{\boldsymbol{X}} + \widetilde{\boldsymbol{N}} \end{aligned}$$

  where $\widetilde{\boldsymbol{X}} = \boldsymbol{F}_n\boldsymbol{x}^{(i)}$ and $\widetilde{\boldsymbol{N}} = \boldsymbol{F}_n\boldsymbol{n}$.
- $\boldsymbol{F}_n$ is unitary: $\widetilde{\boldsymbol{N}} \sim \mathcal{N}_{\mathbb{C}}(\boldsymbol{0}, \sigma^2\boldsymbol{I})$ and $\mathbb{E}[|\widetilde{\boldsymbol{X}}|^2] = \mathbb{E}\left[\left|\boldsymbol{x}^{(i)}\right|^2\right]$.
- OFDM converts the ISI channel into n parallel flat fading channels

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Orthogonal Frequency-Division Multiplexing (OFDM)

- Circulant matrices are diagonalised by the discrete Fourier transform matrix $\boldsymbol{F}_n$, whose elements are

$$F(k,\ell) = \frac{1}{\sqrt{n}} e^{-j2\pi\frac{k\ell}{n}}, \qquad (k = 0, \ldots, n-1, \quad \ell = 0, \ldots, n-1)$$

- We thus have $\widetilde{\boldsymbol{H}} = \boldsymbol{F}_n^H \operatorname{diag}(H_0, \ldots, H_{n-1})\, \boldsymbol{F}_n$, where $H_i = \sum_{\ell=0}^{N_P-1} h_\ell e^{-j2\pi\frac{i\ell}{n}}$ are the DFT coefficients of the rows of $\widetilde{\boldsymbol{H}}$.
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Frequency-Selective Fading

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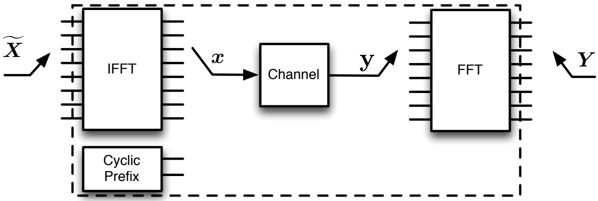
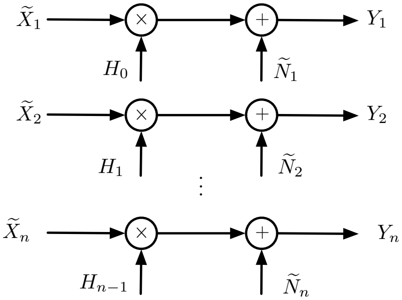
Frequency-Selective Fading

OFDM System Model





Complexity of FFT is $\mathcal{O}(n \log n)$