4F5: Advanced Wireless Communications
Handout 8: Transmission over Fading Channels

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Lent 2012
Underspread Channels

Considered channel models

We consider

1. frequency-flat fading
2. slow frequency-selective fading

Most common scenarios for underspread channels.
Underspread Channels

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Most common scenarios for underspread channels.
Frequency-Flat Fading

Channel Model

Channel output is given by

\[ y_i = h_i x_i + n_i, \quad i = 1, 2, \ldots \]

- Slow fading: \( h_i = h \).
- Fast fading: \( h_1, h_2, \ldots \) are i.i.d.
- Block fading: \( h_i \) is constant for \( T_c \) time slots and changes then independently, i.e.,

\[ y_i = h_{[i/T_c]} x_i + n_i, \quad i = 1, 2, \ldots \]

where \( h_1, h_2, \ldots \) are i.i.d.

- Receiver has perfect knowledge of \( h_1, h_2, \ldots \): the channel can be estimated by transmitting pilot symbols.
- Transmitter has no knowledge of \( h_1, h_2, \ldots \): receiver would need to transfer channel estimates to transmitter.
Frequency-Flat Fading

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Block fading channel

\[ \begin{array}{c}
\bullet \quad \bullet \quad h_1 \\
\bullet \quad h_2 \\
\end{array} \]
Frequency-Flat Fading

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![Block fading channel](image-url)
Frequency-Flat Fading

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Block fading channel

\[ \begin{array}{c}
\vdots
h_1 \hfill h_2 \hfill h_3 \hfill \vdots
\end{array} \]
Uncoded Transmission

Performance Analysis

- Assume that $h_1, h_2, \ldots$ are Gaussian with mean zero and variance 1. This is referred to as Rayleigh fading.
- Consider uncoded BPSK modulation over a Rayleigh fading channel. (Whether the fading is fast or slow does not matter. Why?)
- Let $P_b(h)$ denote the probability of error when $h_i = h$. The average probability of error (averaged over $h_i$) is given by

$$P_b = \int f_H(h)P_b(h)dh$$

- If $|h_i|^2 = 1$ with probability one, then the channel specialises to the AWGN channel for which $P_b^{(G)} = Q\left(\sqrt{2\text{SNR}}\right)$.
- If $|h_i| > 1$ then $P_b(h_i) < P_b^{(G)}$, and if $|h_i| < 1$ then $P_b(h_i) > P_b^{(G)}$. By the convexity of $P_b(h)$ it follows that having a nondeterministic channel gain is detrimental.
- The average probability of error can be computed as

$$P_b = \int f_H(h)Q\left(\sqrt{2|h|^2\text{SNR}}\right)dh = \frac{1}{2} \left(1 - \sqrt{\frac{\text{SNR}}{1 + \text{SNR}}}\right) \approx \frac{1}{4\text{SNR}}$$
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  \[ P_b = \int f_H(h) P_b(h) dh \]
  where $f_H(h)$ is the probability density function of $h$. If $|h_i|^2 = 1$ with probability one, then the channel specialises to the AWGN channel for which $P_b^{(G)} = Q\left(\sqrt{2} \text{SNR}\right)$.
- If $|h_i| > 1$ then $P_b(h_i) < P_b^{(G)}$, and if $|h_i| < 1$ then $P_b(h_i) > P_b^{(G)}$. By the convexity of $P_b(h)$ it follows that having a nondeterministic channel gain is detrimental.
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Assume that $h_1, h_2, \ldots$ are Gaussian with mean zero and variance 1. This is referred to as Rayleigh fading.

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The average probability of error can be computed as

$$P_b = \int f_H(h) Q \left( \sqrt{2|h|^2SNR} \right) dh = \frac{1}{2} \left( 1 - \sqrt{\frac{SNR}{1 + SNR}} \right) \approx \frac{1}{4SNR}$$
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Uncoded Transmission

Probability of Error

\[ P_b \]

\[ \text{SNR (dB)} \]

\[ \text{Rayleigh fading} \]

\[ \text{AWGN} \]

\[ 10^{-6} \]

\[ 10^{-5} \]

\[ 10^{-4} \]

\[ 10^{-3} \]

\[ 10^{-2} \]

\[ 10^{-1} \]

\[ 10^0 \]

\[ 10^1 \]

\[ 10^2 \]

\[ 10^3 \]

\[ 10^4 \]

\[ 10^5 \]

\[ 10^6 \]

\[ 0 \]

\[ 5 \]

\[ 10 \]

\[ 15 \]

\[ 20 \]

\[ 25 \]

\[ 30 \]
Uncoded Transmission

Q-Function is Convex
Diversity

Repetition Code

- Bad error performance is due to events where $h_i$ is small.
- Suppose we transmit the same BPSK symbol twice: we thus observe

$$y_1 = h_1 x + n_1 \quad \text{and} \quad y_2 = h_2 x + n_2$$

- If $h_1$ and $h_2$ are independent, then it is less likely that both $h_1$ and $h_2$ are small.
- Maximum-ratio combining:

$$h_1^* y_1 + h_2^* y_2 = (|h_1|^2 + |h_2|^2) x + h_1^* n_1 + h_2^* n_2$$

maximizes the signal-to-noise ratio at the receiver.

- The probability of error is upper-bounded by

$$\mathbb{E} \left[ Q \left( \sqrt{2 (|h_1|^2 + |h_2|^2) \text{SNR}} \right) \right] \leq \frac{1}{2} \mathbb{E} \left[ \exp \left( - (|h_1|^2 + |h_2|^2) \text{SNR} \right) \right]$$

$$= \frac{1}{2} \mathbb{E} \left[ \exp \left( - |h_1|^2 \text{SNR} \right) \right] \mathbb{E} \left[ \exp \left( - |h_2|^2 \text{SNR} \right) \right]$$

$$= \frac{1}{2(1 + \text{SNR})^2}$$
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  = \frac{1}{2(1 + \text{SNR})^2}
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Diversity

Diversity Techniques

- If the receiver observes \( D \) independently faded replicas of the same information signal, the probability of all deep fades is reduced significantly: \( P_b \propto \text{SNR}^{-D} \)

- Diversity order \( d \triangleq \lim_{\text{SNR} \to \infty} -\frac{\log P_b}{\log \text{SNR}} \) (asymptotic slope of \( P_e \) in a log-log scale)

- There are multiple ways of achieving diversity in practical wireless systems:
  - Frequency: transmitting the same information over \( D \) carriers
  - Time: transmitting the same information over \( D \) time slots
  - Space: equip the receiver, transmitter or both with multiple antennas

- **Note:** time diversity does not require repetition coding. We will see later that usual channel coding works well.

- We have to ensure that we observe *independently faded* replicas of the transmitted signal. To this end, the carriers/time slots/antennas must be spaced sufficiently apart from each other.
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- Diversity order $d \triangleq \lim_{\text{SNR} \to \infty} -\frac{\log P_b}{\log \text{SNR}}$ (asymptotic slope of $P_e$ in a log-log scale)

- There are multiple ways of achieving diversity in practical wireless systems:
  - **Frequency**: transmitting the same information over $D$ carriers
  - **Time**: transmitting the same information over $D$ time slots
  - **Space**: equip the receiver, transmitter or both with multiple antennas

- **Note**: time diversity does not require repetition coding. We will see later that usual channel coding works well.

- We have to ensure that we observe *independently faded* replicas of the transmitted signal. To this end, the carriers/time slots/antennas must be spaced sufficiently apart from each other.
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Diversity

Multiple Receive Antennas

- Suppose we have 1 transmit and \( N_R \) receive antennas. Then,

\[
y_i = h_i x_i + n_i, \quad i = 1, 2, \ldots
\]

- With maximum-ratio combining we obtain

\[
h_i^H y_i = \|h_i\|^2 x_i + h_i^H n_i = \left( |h_i(1)|^2 + \ldots + |h_i(N_R)|^2 \right) x_i + \tilde{n}_i
\]

- The error probability is upper-bounded by

\[
P_b \leq \frac{1}{2 (1 + \frac{\text{SNR}}{N_R})^{N_R}} \propto \text{SNR}^{-N_R}
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Diversity

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Diversity
Probability of Error: Multiple Receive Antennas

\[ P_b = 10^{\frac{-NR}{2}} \]

\[ NR = 1 \quad NR = 4 \]

\[ SNR \ (dB) \]

\[ P_b \]

\[ 10^0 \quad 10^{-1} \quad 10^{-2} \quad 10^{-3} \quad 10^{-4} \quad 10^{-5} \quad 10^{-6} \]

\[ 0 \quad 10 \quad 20 \quad 30 \quad 40 \quad 50 \]

\( N_R = 4 \quad N_R = 1 \)

\( P_b \) vs. SNR (dB) for different values of \( N_R \).
Suppose we have $N_T$ transmit and 1 receive antennas, and assume that the fading is slow. Then,

$$y_i = h^T x_i + n_i, \quad i = 1, 2, \ldots$$

If the transmitter would know $h$, then we could transmit $h^* x_i$ to obtain

$$y_i = h^T h^* x_i + n_i = \left( |h(1)|^2 + \ldots + |h(N_T)|^2 \right) x_i + n_i$$

But (by assumption) the transmitter has no knowledge of $h_i$!

Repetition coding across antennas: $x_1 = (x, 0, \ldots, 0)^T$, $x_2 = (0, x, 0, \ldots, 0)^T$, ...  

The rate of this scheme is $\frac{1}{N_T}$.

For $N_T = 2$, Alamouti’s scheme achieves diversity order 2 at rate 1:

- transmit two symbols $(s_1, s_2)$ over two time slots

$$x_1 = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}, \quad \text{and} \quad x_2 = \begin{pmatrix} -s_2^* \\ s_1^* \end{pmatrix}$$

- The receiver computes

$$h(1)^* y_1 + h(2)^* y_2^* = \left( |h(1)|^2 + |h(2)|^2 \right) s_1 + \tilde{n}_1, \quad \tilde{n}_1 \sim \mathcal{CN} \left( 0, (|h(1)|^2 + |h(2)|^2) \sigma^2 \right)$$

$$h(2)^* y_1 - h(1)^* y_2^* = \left( |h(1)|^2 + |h(2)|^2 \right) s_2 + \tilde{n}_2, \quad \tilde{n}_2 \sim \mathcal{CN} \left( 0, (|h(1)|^2 + |h(2)|^2) \sigma^2 \right)$$
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Diversity
Alamouti’s Scheme

Fig. 4. The BER performance comparison of coherent BPSK with MRRC and two-branch transmit diversity in Rayleigh fading.

Substituting the appropriate equations we have

(15)

These combined signals are then sent to the maximum likelihood decoder which for signal uses the decision criteria expressed in (17) or (18) for PSK signals.

Choose \( s_i \) iff

(17)

Choose \( s_{i-1} \) iff

(18)

Similarly, for using the decision rule is to choose signal

(19)

or, for PSK signals,

choose \( s_i \) iff

(20)

The combined signals in (16) are equivalent to that of four-branch MRRC, not shown in the paper. Therefore, the resulting diversity order from the new two-branch transmit diversity scheme with two receivers is equal to that of the four-branch MRRC scheme.

It is interesting to note that the combined signals from the two receive antennas are the simple addition of the combined signals from each receive antenna, i.e., the combining scheme is identical to the case with a single receive antenna. We may hence conclude that, using two transmit and receive antennas, we can use the combiner for each receive antenna and then simply add the combined signals from all the receive antennas to obtain the same diversity order as -branch MRRC. In other words, using two antennas at the transmitter, the scheme doubles the diversity order of systems with one transmit and multiple receive antennas.

IV. ERROR PERFORMANCE SIMULATIONS

The diversity gain is a function of many parameters, including the modulation scheme and FEC coding. Fig. 4 shows the BER performance of uncoded coherent BPSK for MRRC and the new transmit diversity scheme in Rayleigh fading. It is assumed that the total transmit power from the two antennas for the new scheme is the same as the transmit power from the single transmit antenna for MRRC. It is also assumed that the amplitudes of fading from each transmit antenna to each receive antenna are mutually uncorrelated Rayleigh distributed and that the average signal powers at each receive antenna from each transmit antenna are the same. Further, we assume that the receiver has perfect knowledge of the channel. Although the assumptions in the simulations may seem highly unrealistic, they provide reference performance curves for comparison with known techniques. An important issue is

Diversity

Multiple-Input Multiple-Output (MIMO)

- Suppose we have $N_T$ transmit and $N_R$ receive antennas.
- The output of the multiple-input multiple-output (MIMO) flat fading channel is

$$y_i = H_i x_i + n_i, \quad i = 1, 2, \ldots$$

where

$$x_i = \begin{pmatrix} x_i(1) \\ \vdots \\ x_i(N_T) \end{pmatrix}, \quad y_i = \begin{pmatrix} y_i(1) \\ \vdots \\ y_i(N_R) \end{pmatrix}, \quad n_i = \begin{pmatrix} n_i(1) \\ \vdots \\ n_i(N_R) \end{pmatrix}, \quad H_i = \begin{pmatrix} h_i(1, 1) & \cdots & h_i(N_T, 1) \\ \vdots & \ddots & \vdots \\ h_i(1, N_R) & \cdots & h_i(N_T, N_R) \end{pmatrix}$$

- The diversity order is not larger than $N_T N_R$. 

<Diagram of MIMO system>
Suppose we have $N_T$ transmit and $N_R$ receive antennas.

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Frequency-Selective Fading

Channel Model

- For slow frequency-selective fading, the channel output is

\[ y_i = \sum_{\ell=0}^{N_P-1} h_{i-\ell} x_i + n_i, \quad i = 1, 2, \ldots \]

- Thus, each channel realisation is a random realisation of an ISI channel with \( N_P \) taps having zero mean and variance given by the multipath delay profile \( R_h(\tau) \).
- Since the fading is slow, the taps do not depend on \( i \).

Orthogonal Frequency-Division Multiplexing (OFDM)

- Converts ISI channel into parallel frequency-flat fading channels.
- Transmit block-wise in blocks of \( n \) symbols.
- Before each block introduce a guard period of \( N_P - 1 \).
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  \[ y_i = \sum_{\ell=0}^{N_P-1} h_\ell x_{i-\ell} + n_i, \quad i = 1, 2, \ldots \]

- Thus, each channel realisation is a random realisation of an ISI channel with \( N_P \) taps having zero mean and variance given by the multipath delay profile \( R_h(\tau) \).
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- Converts ISI channel into parallel frequency-flat fading channels.
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- Before each block introduce a guard period of \( N_P - 1 \).
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- Transmit \( n \) information symbols \( \mathbf{x}^{(i)} = (x_1, \ldots, x_n)^T \).
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- Let \( \mathbf{y} = (y_1, \ldots, y_n)^T \), \( \mathbf{n} = (n_1, \ldots, n_n)^T \) and \( \mathbf{x} = (\mathbf{x}^{(p)}, \mathbf{x}^{(i)})^T \).
- In matrix notation

\[
\mathbf{y} = \begin{pmatrix}
    h_{N_P-1} & \cdots & h_0 & 0 & \cdots & \cdots & 0 \\
    0 & h_{N_P-1} & \cdots & h_0 & 0 & \cdots & 0 \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
    0 & \cdots & \cdots & 0 & h_{N_P-1} & \cdots & h_0 \\
\end{pmatrix}
\begin{pmatrix}
    \mathbf{x} \\
\end{pmatrix}
+ \mathbf{n}
\]

- This can be written as \( \mathbf{y} = \tilde{\mathbf{H}}\mathbf{x}^{(i)} + \mathbf{n} \), where

\[
\tilde{\mathbf{H}} = \begin{pmatrix}
    h_0 & 0 & \cdots & 0 & h_{N_P-1} & \cdots & \cdots & h_1 \\
    h_1 & h_0 & 0 & \cdots & 0 & h_{N_P-1} & \cdots & h_2 \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
    0 & \cdots & \cdots & 0 & h_{N_P-1} & \cdots & h_0 \\
\end{pmatrix}
\]

- Note that \( \tilde{\mathbf{H}} \) is a circulant matrix.
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- In matrix notation
  \[
  \mathbf{y} = \begin{pmatrix}
  h_{NP-1} & \cdots & h_0 & 0 & \cdots & \cdots & 0 \\
  0 & h_{NP-1} & \cdots & h_0 & 0 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
  0 & \cdots & \cdots & 0 & h_{NP-1} & \cdots & h_0 \\
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$$F(k, \ell) = \frac{1}{\sqrt{n}} e^{-j2\pi \frac{k\ell}{n}}, \quad (k = 0, \ldots, n-1, \quad \ell = 0, \ldots, n-1)$$

- We thus have $\tilde{H} = F_n^H \text{diag}(H_0, \ldots, H_{n-1}) F_n$, where $H_i = \sum_{\ell=0}^{N_P-1} h\ell e^{-j2\pi \frac{\ell i}{n}}$ are the DFT coefficients of the rows of $\tilde{H}$.

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Frequency-Selective Fading
OFDM System Model

\[
\begin{align*}
\tilde{X} & \xrightarrow{\text{IFFT}} x \xrightarrow{\text{Channel}} y \xrightarrow{\text{FFT}} Y \\
\tilde{X}_1 & \xrightarrow{\times H_0} + \tilde{N}_1 \xrightarrow{Y_1} \\
\tilde{X}_2 & \xrightarrow{\times H_1} + \tilde{N}_2 \\
\vdots & \\
\tilde{X}_n & \xrightarrow{\times H_{n-1}} + \tilde{N}_n \xrightarrow{Y_n}
\end{align*}
\]

Complexity of FFT is $\mathcal{O}(n \log n)$