4F7 Digital Filters and Spectral Estimation
Examples Sheet - Spectrum Estimation

Revision questions

1. Determine the Discrete-time Fourier Transform (DTFT) of the following functions, which have infinite time duration:

   (a) $x_n = \exp(i\omega_0 n)$

   (b) $x_n = \sin(\omega_0 n)$

2. Determine and sketch the magnitude of the DTFT of the following function,

$$x_n = \begin{cases} \exp(in\pi/5), & n = 0, 1, ..., 31 \\ 0, & \text{otherwise} \end{cases}$$

   paying particular attention to central lobe and side lobe characteristics.

3. Determine the power spectrum of the following random processes:

   (a)
$$x_n = A\sin(\omega_0 n + \phi)$$
   where A and ω_0 are constants and ϕ is uniformly distributed between 0 and 2π.

   (b)
$$x_n = A\sin(\omega_0 n + \phi) + v_n$$
   where A and ω_0 are constants and ϕ is uniformly distributed between 0 and 2π and v_n is random white Gaussian noise with variance σ_v^2.

Power Spectrum Estimation

4. Consider a random process $\{X_n\}$ composed of two random phase sine-waves:

$$x_n = A\sin(\omega_1 n + \phi_1) + B\sin(\omega_2 n + \phi_2) + v_n$$

   where A and B are constants, $\omega_2 > \omega_1$, ϕ_1 and ϕ_2 are independent and uniformly distributed between 0 and 2π and v_n is white noise with variance σ_v^2.

   (a) Determine and sketch the power spectrum of the process.

(b) Sketch (approximately) the expected value of the periodogram constructed from N data points measured from the random process.

(c) Two frequency components can be approximately resolved if the centre (or main) lobes of the *expected value* of the periodogram do not overlap. Assuming the main lobe width is c/M (for some constant c) where M is the window length, determine the relationship between the number of samples N used to construct the periodogram and $\omega_2 - \omega_1$ so that these frequencies can be resolved.

5. The *modified* periodogram applies a window to the data before computing the DTFT:

$$\hat{S}_M(e^{j\omega}) = \frac{1}{NU} \left| \sum_{n=0}^{N-1} w_n x_n e^{-jn\omega} \right|^2$$

where $U = \frac{1}{N} \sum_{n=0}^{N-1} |w_n|^2$.

(a) Show that the expected value of the modified periodogram is:

$$E[\hat{S}_M(e^{j\omega})] = \frac{1}{NU} \sum_{k=-(N-1)}^{+N-1} v_k R_{XX}[k] e^{-jk\omega}$$

where

$$v_k = (\{w_n\} * \{w_{-n}\})(k)$$

i.e. the convolution of w_k with itself time-reversed.

(b) Comment on the relationship of this result with the expected value of the standard periodogram and discuss how the modified periodogram might achieve a different trade-off between frequency resolution and variance of the estimate.

6. A stationary random phase complex exponential is given by

$$x_n = \exp(i(n\omega_0 + \phi))$$

where ϕ is uniformly distributed between 0 and 2π.

(a) What is the power spectrum for this process? (For a complex process, the autocorrelation function is defined as $R_{XX}[k] = E[x_n^* x_{n+k}]$, and the power spectrum is the DTFT of R_{XX}.)

(b) Write an expression for the periodogram estimate for a sample of N data points measured from the process.

(c) Hence determine the mean and variance of the periodogram for this process. Does this agree with the 'rule of thumb' that the variance of the periodogram is approximately equal to the true power spectrum squared? If not, why is it that this process could be different from the rule?

7. (a) State the variance of periodogram power spectral estimates of white Gaussian noise having variance σ^2. Comment on the significance of this result for power spectrum estimation of noise-like processes.

(b) The Bartlett procedure segments the available data into K contiguous subsequences of length N_B and computes a spectral estimate from:

$$\hat{S}_X(e^{j\omega}) = \frac{1}{K}\sum_{k=0}^{K-1} \hat{S}_X^{(k)}(e^{j\omega})$$

where $S_X^{(k)}(e^{j\omega})$ is the periodogram of the kth subsequence.
Show that the Bartlett procedure reduces the variance of the spectral estimate of white noise by K times.

(c) For general signals, show that the Bartlett procedure is biased as for the periodogram but asymptotically unbiased.

(d) Show that the frequency resolution of the Bartlett method is K times worse than that of the periodogram applied to the same data overall length.

Parametric Methods

8. Estimates are made of the correlation function of a particular signal and the values obtained are:

$$R_{XX}[0] = 7.24$$
$$R_{XX}[1] = 3.6$$

Determine the parameter values of the 1st order MA model

$$H(z) = b_0 + b_1 z^{-1}$$

which matches these correlation by:
(a) Directly solving of the MA equations

$$\begin{bmatrix} R_{XX}[0] \\ R_{XX}[1] \\ \vdots \\ R_{XX}[Q] \end{bmatrix} = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_Q \end{bmatrix}$$

where

$$c_r = \begin{cases} \sum_{q=r}^{Q} b_q\, b_{q-r} & , \quad r \le Q \\ 0 & , \quad r > Q \end{cases}$$

(b) By spectral factorisation.

Sketch the power spectral estimate obtained using this MA model.

9. Fit a 1st order AR model

$$H(z) = \frac{1}{a_0 + a_1 z^{-1}}$$

to the correlation data given in the previous question and sketch the resulting spectral estimate. Do you have any reason to suppose that this estimate is better than that obtained using the MA model?

10. The ARMA(P,Q) model is

$$x_n = -\sum_{p=1}^{P} a_p x_{n-p} + \sum_{q=0}^{Q} b_q w_{n-q}$$

where $\{w_n\}$ is a white noise sequence with mean zero and unit variance. The estimated autocorrelation function of $\{x_n\}$ at lags $k = 0, 1, 2, 3, 4$ are

$$\widehat{R}_{XX}[0] = 2, \quad \widehat{R}_{XX}[1] = 1, \quad \widehat{R}_{XX}[2] = -1, \quad \widehat{R}_{XX}[3] = 0.5, \quad \widehat{R}_{XX}[4] = 0.$$

For $P = 1$ and $Q = 1$, estimate a_1, b_0 and b_1 using these $\widehat{R}_{XX}$ values. Why would you not consider the model ARMA(0,1)?

11. (Computer exercise) Consider the autoregressive random process

$$x_n = -a_1 x_{n-1} - a_2 x_{n-2} + b_0 w_n$$

where w_n is zero mean unit variance white noise.
(a) With $a_1 = 0$, $a_2 = 0.81$ and $b_0 = 1$ generate 24 samples of the random process x_n.
(b) Estimate the autocorrelation sequence using the biased (and unbiased) estimate in the lecture notes and compare it to the true autocorrelation sequence.
(c) Using your estimated autocorrelation sequence, estimate the power spectrum of x_n by computing the Fourier transform of $\widehat{R}_{XX}$. (Hint: periodogram)
(d) Using the estimate $\widehat{R}_{XX}$ from (b), use the Yule-Walker equations to estimate a_1, a_2 and b_0 and comment on the accuracy of your estimates.
(e) Estimate the power spectrum using the estimated values from (d) as follows:

$$\widehat{S}_X(e^{j\omega}) = \frac{b_0^2}{|1 + a_1 e^{-j\omega} + a_2 e^{-2j\omega}|^2}$$

(f) Compare your power spectrum estimates with the true power spectrum. Repeat the above experiment with more data, i.e. more than 24 points.

Suitable past tripos questions: years 2010–2016. (Note that only topics covered in lectures will be assessed.)

Worked solutions

Q1 a)

$$\begin{aligned} X(e^{i\omega}) &= \sum_{n=-\infty}^{+\infty} x_n e^{-in\omega} \\ &= \sum_{n=-\infty}^{+\infty} \exp(i\omega_0 n) e^{-in\omega} \\ &= \sum_{n=-\infty}^{+\infty} \exp(i(\omega_0 - \omega)n) \\ &= 2\pi \sum_{m=-\infty}^{+\infty} \delta(\omega - \omega_0 + 2\pi m) \end{aligned}$$

b) Use

$$\sin(\omega_0 n) = \frac{1}{2j}(\exp(i\omega_0 n) - \exp(-i\omega_0 n))$$

and the answer to Q1a.

Q2

$$\begin{aligned} X(e^{i\omega}) &= \sum_{n=-\infty}^{+\infty} x_n e^{-in\omega} \\ &= \sum_{n=0}^{N-1} \exp(in(\omega_0 - \omega)) \\ &= \frac{1 - \exp(i(\omega_0 - \omega))^N}{1 - \exp(i(\omega_0 - \omega))} \\ &= \exp(i(N-1)(\omega_0 - \omega)/2) \frac{\sin((\omega_0 - \omega)N/2)}{\sin((\omega_0 - \omega)/2)} \end{aligned}$$

and

$$|X(e^{i\omega})| = \left| \frac{\sin((\omega_0 - \omega)N/2)}{\sin((\omega_0 - \omega)/2)} \right|$$

with $\omega_0 = 0.2\pi$, see figure.

Q3 (a) Standard material - see e.g. 3F3 lecture notes. (b) Get this by noting that the sine and noise terms are uncorrelated. Hence you can calculate the power spectrum of each term and add them together to get the result.

Q4 a) From Q3 we have the power spectrum of a single random phase sine wave in noise. To get the two-sine version, notice that both sine terms and the noise

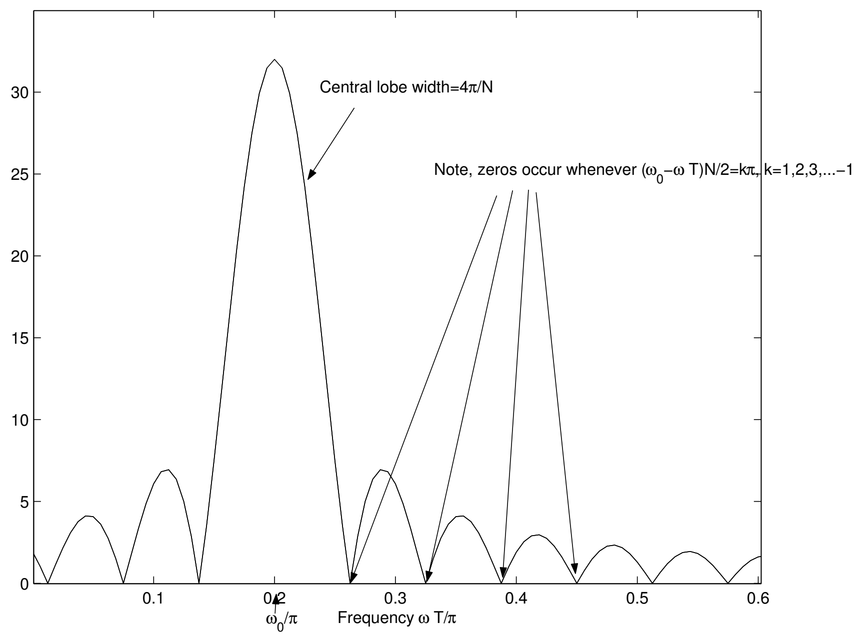

Figure 1: plot of $|X(e^{i\omega T})| = \left|\frac{\sin((\omega_0 - \omega T)N/2)}{\sin((\omega_0 - \omega T)/2)}\right|$

term are mutually uncorrelated (check this if you are unsure). Hence to get overall power spectrum, just add together the power spectra of the sine waves with that of the noise (white).

b) Expected value of the periodogram is (see lecture notes):

$$\boxed{E[\hat{S}_X(e^{j\omega})] = \frac{1}{2\pi}\int_{-\pi}^{\pi} W(e^{j\theta}) S_X(e^{j(\omega-\theta)})\, d\theta} \tag{1}$$

i.e. the convolution of the true power spectrum with the spectrum of the window. The convolution is easy to sketch since the power spectrum is a train of delta functions plus a noise floor. From lectures, $W(e^{j\omega}) = (1/N)\,(\sin(N\omega/2)/\sin(\omega/2))^2$.

c) Need $\omega_2 - \omega_1 > c/M$ where $M = 2N - 1$.

Q5 a)

$$
\begin{aligned}
E[\hat{S}_M(e^{j\omega})] &= \frac{1}{NU} E \left| \sum_{n=0}^{N-1} w_n x_n e^{-jn\omega} \right|^2 \\
&= \frac{1}{NU} E[\sum_{n=-\infty}^{\infty} w_n x_n e^{-jn\omega} \sum_{m=-\infty}^{\infty} w_m x_m e^{+jm\omega}] \\
&= \frac{1}{NU} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} w_n w_m E[x_n x_m] e^{-j(n-m)\omega} \\
&= \frac{1}{NU} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} w_n w_m R_{XX}[n-m] e^{-j(n-m)\omega} \\
&= \frac{1}{NU} \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} w_n w_{n-k} R_{XX}[k] e^{-jk\omega} \quad \text{with } k = n-m \\
&= \frac{1}{NU} \sum_{k=-\infty}^{\infty} \{ \sum_{n=-\infty}^{\infty} w_n w_{n-k} \} R_{XX}[k] e^{-jk\omega} \\
&= \frac{1}{NU} \sum_{k=-\infty}^{\infty} v_k R_{XX}[k] e^{-jk\omega}
\end{aligned}
$$

b) Hence

$$
E[\hat{S}_M(e^{j\omega})] = \frac{1}{2\pi NU} S_X(e^{j\omega}) * |W(e^{j\omega})|^2.
$$

Modified periodogram allows choice of a window function with suitable spectral leakage and spectral smearing properties to the application. This contrasts with the periodogram, in which the windowing function is fixed as the rectangular window - narrow central lobe but very severe sidelobes.

Q6 a) Power spectrum is a train of delta functions centred at frequency ω_0:

$$
S(e^{i\omega}) = 2\pi \sum_{n=-\infty}^{+\infty} \delta(\omega - \omega_0 + 2\pi n)
$$

b) The periodogram is

$$
\begin{aligned}
N \times \hat{S}(e^{i\omega}) &= | \sum_{n=0}^{N-1} x_n \exp(-in\omega)|^2 \\
&= | \sum_{n=0}^{N-1} \exp(i(n\omega_0 + \phi)) \exp(-in\omega)|^2 \\
&= | \exp(i\phi)|^2 | \sum_{n=0}^{N-1} \exp(-in(\omega - \omega_0))|^2 \\
&= | \sum_{n=0}^{N-1} \exp(-in(\omega - \omega_0))|^2
\end{aligned}
$$

(Further simplification unnecessary for this question.) The important point here for the next parts is that the periodogram estimate does not depend on the value of the random variable ϕ. Hence the variance of the periodogram estimate is zero, see next part.

c) The mean is the rectangular window spectrum shifted across in frequency to center frequency ω_0. The variance is, however, zero. Thus in this case the variance is not the rule of thumb. i.e. for single complex exponentials the periodogram gives no variability. This is because the periodogram of a complex exponential is constant whatever the phase of the exponential

Q7 a) Variance of periodogram is approximately σ^4 for all data lengths, becoming exact as data length goes to infinity. This means that the variance does not decrease as data length increases. We can expect the periodogram of other non-Gaussian noise processes to behave similarly.

b) For white Gaussian noise, each of the K subsequences of data are statistically independent so that the periodogram estimates for each subsequence are also statistically independent. Consider a particular frequency component $\hat{S}_X^{(k)}(e^{j\omega})$ from each of the K periodograms. In order to ease the notation, let

$$Z_k \equiv \hat{S}_X^{(k)}(e^{j\omega_i T}).$$

By independence

$$\text{var}\left(\frac{1}{K}\sum_{k=1}^{K} Z_k\right) = \frac{1}{K^2}\sum_{k=1}^{K} \text{var}\left(Z_k\right)$$

i.e. the variance of the spectrum estimate has been reduced by a factor of K. To be more precise,

$$\text{var}(Z_k) = \sigma^4\left(1 + \left\{\frac{\sin(N_B\omega)}{N_B\sin(\omega)}\right\}^2\right)$$

assuming $N = N_B K$, with N being the total number of data points available. When the data is not segmented ($K = 1$) the variance has the same expression except that N_B should be replaced by N. The ratio of these two quantities tends to $1/K$ as N increases.

c) $E[\hat{S}_X(e^{j\omega})] = E[\frac{1}{K}\sum_{k=1}^{K}\hat{S}_X^{(k)}(e^{j\omega})]$. Each expectation term is an expectation of a periodogram estimate for each sub-block. Hence, as for the periodogram the method is biased but asymptotically unbiased.

d) Clearly each periodogram in the summation corresponds to a data window length $N_B = N/K$, where N is the total number of data points available. (Note the value of the window length in the expected value of the periodogram is $2\times$datalength-1) Hence each periodogram estimate in the Bartlett summation has K times poorer resolution, since central lobe of window spectrum is K times wider than that of the full perdiodogram estimate for all N data points.

Q8 Either method should give:

$$b_0 = 2.0 \qquad\qquad b_1 = 1.8$$

Remember that the spectral factorisation method only gives the roots of the polynomial and the scaling must be calculated separately.

(a) By direct solution ($Q = 1$)

$$\begin{bmatrix} R_{XX}[0] \\ R_{XX}[1] \end{bmatrix} = \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} \sum_{q=0}^{1} b_q^2 \\ \sum_{q=1}^{1} b_q b_{q-1} \end{bmatrix} = \begin{bmatrix} b_0^2 + b_1^2 \\ b_1 b_0 \end{bmatrix}.$$

Inserting values

$$\begin{bmatrix} 7.24 \\ 3.6 \end{bmatrix} = \begin{bmatrix} b_0^2 + b_1^2 \\ b_1 b_0 \end{bmatrix}$$

Solve to get

$$\begin{aligned} b_1 &= 3.6/b_0 \\ 7.24 &= b_0^2 + \frac{3.6^2}{b_0^2} \\ b_0^4 &- 7.24 b_0^2 + 3.6^2 = 0 \end{aligned}$$

$(b_0^2 - 4)(b_0^2 - 3.24) = 0$. For $b_0 = \pm 2$, $b_1 = \pm 1.8$ and so two possible MA models.

(b) By spectral factorization: first solve for the zeros of $\sum_{r=-Q}^{Q} R_{XX}[r] z^{-r}$.

$$\begin{aligned} & R_{XX}[-1]z + R_{XX}[0] + R_{XX}[1]z^{-1} \\ = \; & 3.6z + 7.24 + 3.6z^{-1} \\ = \; & z\left(3.6 + 7.24z^{-1} + 3.6z^{-2}\right) \\ = \; & z\left(z^{-1} + \frac{7.24 + 0.76}{7.2}\right)\left(z^{-1} + \frac{7.24 - 0.76}{7.2}\right) \\ = \; & \left(1 + \frac{8}{7.2}z\right)\left(z^{-1} + \frac{6.48}{7.2}\right) \end{aligned}$$

The root in the unit circle is $-7.2/8 = -0.9$.

Now $g\left(1 - z^{-1}(-0.9)\right)$ is to be solved for g.

$$\begin{aligned} c_0 &= g^2 + 0.9^2 g^2 \\ 7.24 &= g^2(1 + 0.9^2) \\ g &= \sqrt{\frac{7.24}{1 + 0.9^2}} = 2 \end{aligned}$$

So the MA model is $B(z) = 2 + 1.8z^{-1}$, or $b_0 = 2, b_1 = 1.8$. The MA spectrum corresponding to the model is shown in figure 2

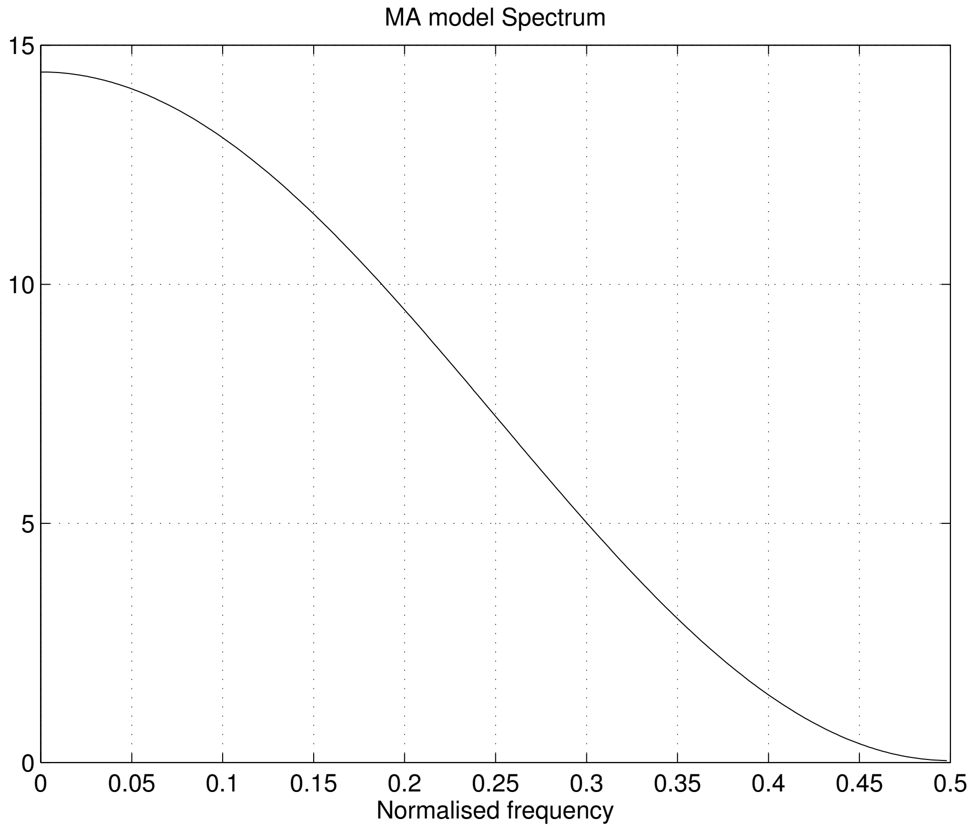

Figure 2:

Q9 The AR coefficients are:

$$b_0 = 2.3345 \qquad a_0 = 1 \qquad a_1 = -0.4972$$

To fit the AR model use the Yule-Walker equations P=1,Q=0:

$$\begin{aligned} R_{XX}[0]a_1 &= R_{XX}[-1] \\ a_1 &= \frac{-3.6}{7.24}. \end{aligned}$$

Now solve for b_0:

$$\begin{aligned} b_0^2 &= [\ R_{XX}[0] \quad R_{XX}[-1]\ ][\ 1 \quad a_1\ ]^{\mathrm{T}} \\ &= R_{XX}[0] - \frac{R_{XX}[1]^2}{R_{XX}[0]} = 2.335^2 \end{aligned}$$

$$H(z) = \frac{b_0}{1 + a_1 z^{-1}} = \frac{2.335}{1 - \frac{3.6}{7.24} z^{-1}}.$$

Power spectrum is $S(e^{j\omega T}) = |H(e^{j\omega T})|^2$. The AR spectrum corresponding to the model is shown in figure 3

Without any prior knowledge of the physical system which produced the signals, one spectral estimate should not be preferred over the other. However, the 1st order MA model assumes that the signal correlation is zero for lags greater than 1

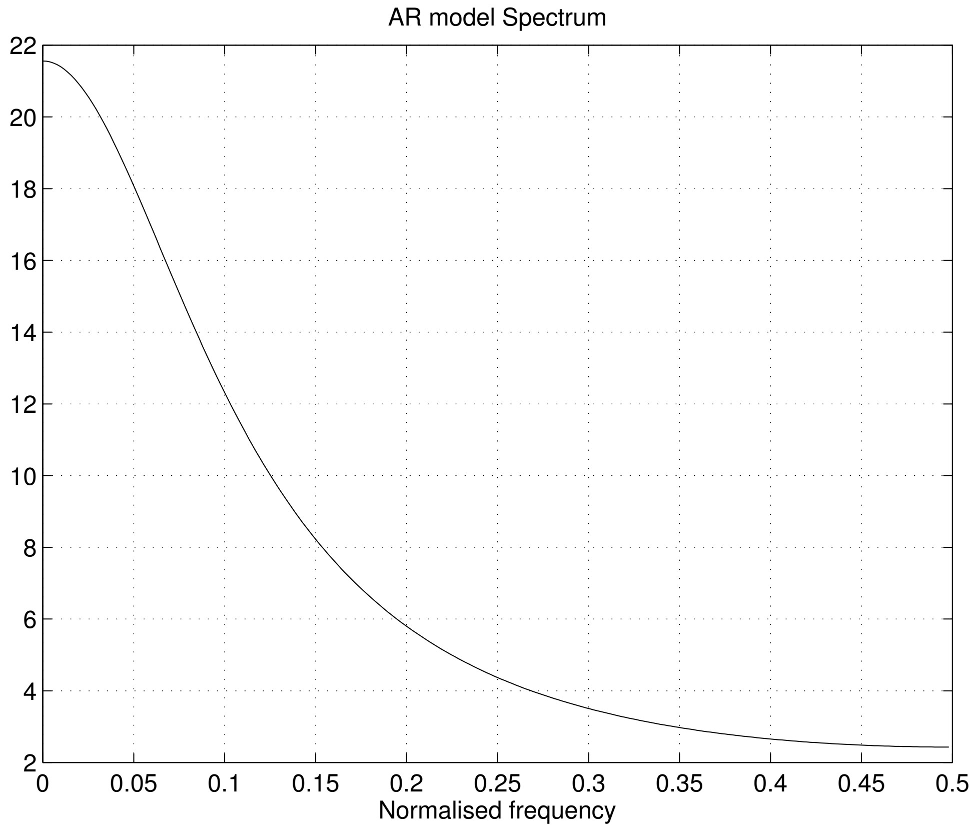

Figure 3:

whereas the AR model assumes that the correlation function satisfies the AR difference equation so that the correlation function is not zero for lags greater than 1. It might be argued that this is a more reasonable reflection of what might be the case in the system which generated the signal.

Q10 The signal model is $x_n = -a_1 x_{n-1} + b_0 w_n + b_1 w_{n-1}$. The Yule-Walker equations are $c_0 = b_0 h_0 + b_1 h_1$, $c_1 = b_1 h_0$ and

$$\begin{bmatrix} R_{XX}[0] + a_1 R_{XX}[-1] \\ R_{XX}[1] + a_1 R_{XX}[0] \\ R_{XX}[2] + a_1 R_{XX}[1] \end{bmatrix} = \begin{bmatrix} b_0 h_0 + b_1 h_1 \\ b_1 h_0 \\ 0 \end{bmatrix}$$

Use the third equation to solve for a_1 to get $a_1 = -\widehat{R}_{XX}[2]/\widehat{R}_{XX}[1] = 1$.

Note now we have 2 equations and unknowns b_0, b_1, h_0, h_1. Observe that

$$y_n = x_n + a_1 x_{n-1}$$

is an ARMA(0,1) model and find its coefficients. We need $R_{YY}[0]$ and $R_{YY}[1]$.

$$\begin{aligned} R_{YY}[0] &= E(y_n y_n) \\ &= E\left(x_n x_n + a_1^2 x_{n-1} x_{n-1} + 2a_1 x_n x_{n-1}\right) \\ &= 2R_{XX}[0] + 2R_{XX}[1] \end{aligned}$$

Similarly

$$\begin{aligned} R_{YY}[1] &= E(y_{n+1}y_n) \\ &= E\left(x_{n+1}x_n + a_1^2 x_n x_{n-1} + a_1 x_{n+1}x_{n-1} + a_1 x_n x_n\right) \\ &= R_{XX}[0] + 2R_{XX}[1] + R_{XX}[2] \end{aligned}$$

So $\widehat{R}_{YY}[0] = 6$ and $\widehat{R}_{YY}[1] = 3$

Now factorise the polynomial $\sum_{r=-Q}^{Q} \widehat{R}_{YY}[r]z^{-r} = 3z + 6 + 3z^{-1} = 3(1+z)(1+z^{-1})$

Take the root $z = -1$. (Nothing within the unit circle in this example.) Now write $B(z) = g(1 - z^{-1}n_1)$ where $n_1 = -1$ is the chosen root.

The constant g is

$$\sqrt{\frac{\widehat{R}_{YY}[0]}{1 + (-n_1)^2}} = \sqrt{3}$$

The MA model parameters are: $b_0 = g = \sqrt{3}$, $b_1 = g \times (-n_1) = \sqrt{3}$.

The ARMA(0,1) will give an autocorellation value of 0 for lags larger than 1 which is inconsistent with the data. So the ARMA(1,1) model is preferable.

Q11 See Matlab code on course webpage.

S.S. Singh
(Previous versions: Simon Godsill, Peter Rayner)