Revision questions - DTFT, DFT and FFT, power spectrum [many of these results are required for answering later questions]

1. Determine the Discrete-time Fourier Transform (DTFT) of the following functions, which have infinite time duration:
   
   (a) \( x_n = \exp(i\omega_0 n) \)
   
   (b) \( x_n = \sin(\omega_0 n) \)

2. (a) Determine and sketch the magnitude of the DTFT of the following function,

   \[ x_n = \begin{cases} 
   \exp(in\pi/5), & n = 0, 1, ..., 31 \\
   0, & \text{otherwise} 
   \end{cases} \]

   paying particular attention to central lobe and side lobe characteristics

   (b) Hence sketch the magnitude of the DTFT of

   \[ x_n = \begin{cases} 
   \sin(n\pi/5), & n = 0, 1, ..., 31 \\
   0, & \text{otherwise} 
   \end{cases} \]

   over the frequency range \( \omega T \in [-2\pi, +2\pi] \).

3. Determine the power spectrum of the following random processes:

   (a) \( x_n = A \sin(\omega_0 n + \phi) \)

   where \( A \) and \( \omega_0 \) are constants and \( \phi \) is uniformly distributed between 0 and \( 2\pi \).

   (b) \( x_n = A \sin(\omega_0 n + \phi) + v_n \)

   where \( A \) and \( \omega_0 \) are constants and \( \phi \) is uniformly distributed between 0 and \( 2\pi \) and \( v_n \) is random white Gaussian noise with variance \( \sigma_v^2 \).
4. The Discrete Fourier Transform (DFT) of a data sequence $x_n$ is defined by:

$$X_p = \sum_{n=0}^{N-1} x_n e^{-j \frac{2\pi}{N} np}$$

$$x_n = \frac{1}{N} \sum_{p=0}^{N-1} X_p e^{j \frac{2\pi}{N} np}$$

Show that:

(a) $x_{-n} = x_{N-n}$, ie. periodic data
(b) $X_{-p} = X_{N-p}$, ie. periodic spectrum
(c) $x_{n-q} \Leftrightarrow e^{-j \frac{2\pi}{N} pq} X_p$, ie. shift theorem
(d) $X_p^{(1)} X_p^{(2)} \Leftrightarrow \sum_{q=0}^{N-1} x_q^{(1)} x_{n-q}^{(2)}$ ie. Circular convolution

5. Calculate and roughly sketch the 16 point DFT of the data sequence:

$$x_n = e^{j \frac{2\pi}{N} kn}$$

for discrete frequency $k = 3.0$ and also $k = 3.5$.

Comment on the significance of the form of the spectra. You may find it helpful to sketch the real (or imaginary) part of the signal $x_n$ and bear in mind the result proved in question 4(a). You may also find it useful (but not necessary) to use MATLAB.

6. Most realisations of the FFT algorithm, in either software or hardware, are designed to deal with complex data so that the algorithm can be used for both forward and inverse transforms. However in many applications the time domain data are real so that there would appear to be some computational inefficiency in using the complex data algorithm.

Show that the DFT of 2 real data sequences $x_n^{(1)}$ and $x_n^{(2)}$ may be computed with a single complex DFT realisation by forming the complex data signal:

$$x_n = x_n^{(1)} + j x_n^{(2)}$$

and that the spectra of the individual signals are given by:

$$X_p^{(1)} = \frac{1}{2} [X_p + X_{N-p}^*]$$

$$X_p^{(2)} = \frac{1}{2j} [X_p - X_{N-p}^*]$$

7. Show that the results from the previous question can be used to compute the DFT of a single signal with $2N$ real data points. Calculate the improvement in computational efficiency compared with using the direct complex-data DFT on the $2N$ real data points.
Windowing and Frequency Resolution

8. The Fourier transform of a continuous-time signal $g(t)$ is to be estimated by evaluating the Fourier transform of the signal viewed through a rectangular window $w(t)$ of duration $T_w$ seconds. Show that the estimated spectrum $G_w(\omega)$ is given by the convolution of the spectrum $G(\omega)$ of the infinite duration signal and the spectrum of the window function $W(\omega)$.

If the signal $g(t)$ is given by:

$$g(t) = a_1 \cos(\omega_1 t) + a_2 \cos(\omega_2 t)$$

sketch the spectrum of $G_w(\omega)$ and roughly estimate the minimum window duration $T_w$ if the two frequencies are to be resolved. You may assume that two frequency components can be approximately resolved if their centre lobes do not overlap, i.e. the 3dB central lobe bandwidth of one component does not overlap with that of the other component.

Parameter values are:

$$a_1 = a_2 = 1$$
$$\omega_1 = 2\pi \times 10^3 \text{ rad.s}^{-1}$$
$$\omega_2 = 2\pi \times 1.1 \times 10^3 \text{ rad.s}^{-1}$$

9. In some practical situations it is necessary to evaluate efficiently the DFT of $M$ data points where $M$ is not a highly composite number so that an FFT algorithm cannot be used directly. A commonly used technique to overcome this problem is known as zero-padding. The principle is to append zero amplitude signal samples to the data to give a total of $N$ data points where $N$ is a highly composite number (usually $N = 2^K$). The DFT of the padded sequence can be computed with an FFT algorithm.

The spectral spacing of the DFT components will be $\frac{2\pi}{N}$ whereas the spectral spacing of the DFT of the unpadded data sequence will be $\frac{2\pi}{M}$, $M < N$. It is sometimes claimed that zero-padding increases the frequency resolution. By considering the DFT as evaluation of the Discrete Time Fourier Transform (DTFT) at a set of discrete frequencies, show that this claim is incorrect.
Non-parametric Power spectrum Estimation

<table>
<thead>
<tr>
<th>Window</th>
<th>Sidelobe level, dB</th>
<th>3dB Bandwidth</th>
<th>6dB bandwidth</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rectangular</td>
<td>-13</td>
<td>0.89(2π/N)</td>
<td></td>
</tr>
<tr>
<td>Bartlett</td>
<td>-27</td>
<td>1.28(2π/N)</td>
<td>1.78(2π/N)</td>
</tr>
<tr>
<td>Hanning</td>
<td>-32</td>
<td>1.44(2π/N)</td>
<td></td>
</tr>
<tr>
<td>Hamming</td>
<td>-43</td>
<td>1.30(2π/N)</td>
<td></td>
</tr>
<tr>
<td>Blackman</td>
<td>-58</td>
<td>1.68(2π/N)</td>
<td></td>
</tr>
</tbody>
</table>

Table 1: Table of properties for discrete time windows. 3dB Bandwidths are measured in normalised frequencies ωT from the middle of the central lobe to the half power point.

10. The table above gives the basic properties of a few standard window functions, with bandwidths stated in terms of normalised frequency ωT.

Consider a random process \{X_n\} composed of two random phase sine-waves:

\[ x_n = A \sin(\omega_1 n + \phi_1) + B \sin(\omega_2 n + \phi_2) + v_n \]

where \(A\) and \(B\) are constants, \(\phi_1\) and \(\phi_2\) are independent and uniformly distributed between 0 and 2π, and \(v_n\) is white noise with variance \(\sigma_v^2\).

(a) Determine and sketch the power spectrum for the process. Hence sketch approximately the expected value of the periodogram for \(N\) data points measured from such a random process.

(b) Determine the approximate data length required for the periodogram to resolve the two frequencies reliably if \(\omega_2 - \omega_1 \geq 0.01\pi T^{-1}\) where \(T\) is the sampling period. You may assume that two frequency components can be approximately resolved if their centre lobes in the expected value of the periodogram do not overlap, i.e. the 6dB central lobe bandwidth of one component does not overlap with that of the other component in the periodogram.

11. The modified periodogram applies a window to the data before computing the DTFT:

\[ \hat{S}_M(e^{j\omega T}) = \frac{1}{NU} \left| \sum_{n=0}^{N-1} w_n x_n e^{-jn\omega T} \right|^2 \]

where \(U = \frac{1}{N} \sum_{n=0}^{N-1} |w_n|^2\).

(a) Show that the expected value of the modified periodogram is:

\[ E[\hat{S}_M(e^{j\omega T})] = \frac{1}{NU} \sum_{k=-\infty}^{+N-1} w_{Mk} R_{XX}[k] e^{-jk\omega T} \]

where

\[ w_{Mk} = w_k \ast w_{-k} \]

i.e. the convolution of \(w_k\) with itself time-reversed. Hence show that

\[ E[\hat{S}_M(e^{j\omega T})] = \frac{1}{2\pi NU} S_X(e^{j\omega T}) \ast |W(e^{j\omega T})|^2 \]

where \(W(e^{j\omega T})\) is the DTFT of the window function \(w_n\).
(b) Comment on the relationship of this result with the expected value of the standard periodogram and discuss how the modified periodogram might achieve a different trade-off between frequency resolution and variance of the estimate.

12. A stationary random phase complex exponential is given by

\[ x_n = \exp(i(n\omega_0 T + \phi)) \]

where \( \phi \) is uniformly distributed between 0 and \( 2\pi \).

(a) What is the power spectrum for this process? [Note that for a complex process, the autocorrelation function is defined as \( R_{XX}[k] = E[x^*_n x_{n+k}] \), and the power spectrum is the DTFT of \( R_{XX} \).]

(b) Write an expression for the periodogram estimate for a sample of \( N \) data points measured from the process.

(c) Hence determine the mean and variance of the periodogram for this process. Does this tally with the ‘rule of thumb’ that the variance of the periodogram is approximately equal to the true power spectrum squared? If not, why is it that this process could be different from the rule?

13. (a) State the variance of periodogram power spectral estimates of white Gaussian noise having variance \( \sigma^2 \). Comment on the significance of this result for power spectrum estimation of noise-like processes.

(b) The Bartlett procedure segments the available data into \( K \) contiguous subsequences of length \( N_B \) and computes a spectral estimate from:

\[
\hat{S}_X(e^{j\omega T}) = \frac{1}{K} \sum_{k=0}^{K-1} \hat{S}_X^{(k)}(e^{j\omega T})
\]

where \( \hat{S}_X^{(k)}(e^{j\omega T}) \) is the periodogram of the \( k \)th subsequence. Show that the Bartlett procedure reduces the variance of the spectral estimate of white noise by \( K \) times.

(c) Show, for general signals, that the Bartlett procedure is biased as for the periodogram, but asymptotically unbiased.

(d) Show that the frequency resolution of the Bartlett method is \( K \) times worse than that of the periodogram applied to the same data overall length.

Parametric Methods

14. Show that if input of linear time invariant discrete-time system is wide-sense stationary **white noise** then the output is also wide-sense stationary, provided the linear system is stable. Does the result still apply if the linear system is unstable? Use this result to derive the autocorrelation function for a stable ARMA process, carefully stating any assumptions required.
15. Estimates are made of the correlation function of a particular signal and the values obtained are:

\[ R_{XX}[0] = 7.24 \]
\[ R_{XX}[1] = 3.6 \]

Determine the parameter values of the 1st order MA model:

\[ H(z) = b_0 + b_1 z^{-1} \]

which matches these correlation values using:

(a) Direct solution of the MA equations:

\[
\begin{bmatrix}
R_{XX}[0] \\
R_{XX}[1] \\
\vdots \\
R_{XX}[Q]
\end{bmatrix}
= 
\begin{bmatrix}
c_0 \\
c_1 \\
\vdots \\
c_Q
\end{bmatrix}
\]

where:

\[
c_r = \begin{cases} 
\sum_{q=r}^{Q} b_q b_{q-r} & , \quad r \leq Q \\
0 & , \quad r > Q
\end{cases}
\]

(b) By spectral factorisation

Sketch the power spectral estimate obtained using this MA model.

16. Fit a 1st order AR model

\[ H(z) = \frac{1}{a_0 + a_1 z^{-1}} \]

to the correlation data given in the previous question and sketch the resulting spectral estimate. Do you have any reason to suppose that this estimate is better than that obtained using the MA model?

17. (Computer exercise) Consider the autoregressive random process

\[ x_n = -a_1 x_{n-1} - a_2 x_{n-2} + b_0 w_n \]

where \( w_n \) is zero mean unit variance white noise.

(a) With \( a_1 = 0, a_2 = 0.81 \) and \( b_0 = 1 \) generate 24 samples of the random process \( x_n \).

(b) Estimate the autocorrelation sequence using the biased (and unbiased) estimate in the lecture notes and compare it to the true autocorrelation sequence.

(c) Using your estimated autocorrelation sequence, estimate the power spectrum of \( x_n \) by computing the Fourier transform of \( \hat{R}_{XX} \). (Hint: periodogram)

(d) Using the estimate \( \hat{R}_{XX} \) from (b), use the Yule-Walker equations to estimate \( a_1, a_2 \) and \( b_0 \) and comment on the accuracy of your estimates.

(e) Estimate the power spectrum using the estimated values from (d) as follows:

\[ \hat{S}_X(e^{j\omega}) = \frac{b_0^2}{\left|1 + a_1 e^{-j\omega} + a_2 e^{-2j\omega}\right|^2} \]

(f) Compare your power spectrum estimates with the true power spectrum. Repeat the above experiment with more data, i.e. more than 24 points.
Suitable past tripos questions:
Most questions from the old I7 course and all questions from current 4F7, including:
4F7 2004 - all questions
4F7 2003 - all questions
I7 2002 - all questions (Q4. quite a challenge!)
I7 2001 - all questions
I7 pre-2001 - most questions, but not questions on the MUSIC algorithm for frequency estimation.

Answers

1. (a) \[ 2\pi \sum_{m=-\infty}^{+\infty} \delta(\omega T + 2\pi m - \omega_0) \]
   (b) \[ \frac{\pi}{j} \sum_{m=-\infty}^{+\infty} \delta(\omega T + 2\pi m - \omega_0) - \delta(\omega T + 2\pi m + \omega_0) \]

2. (a) \[ |X(e^{j\omega T})| = \left| \frac{\sin((0.2\pi - \omega T)N/2)}{\sin((0.2\pi - \omega T)/2)} \right| \]

3. (a) \[ S_X(e^{j\Omega}) = \pi/2A^2 \sum_{n=-\infty}^{+\infty} [\delta(\Omega - \omega_0 - 2n\pi) + \delta(\Omega + \omega_0 - 2n\pi)] \]
   (expressed in terms of normalised frequency \( \Omega = \omega T \))
   (b) \[ S_X(e^{j\Omega}) = \pi/2A^2 \sum_{n=-\infty}^{+\infty} [\delta(\Omega - \omega_0 - 2n\pi) + \delta(\Omega + \omega_0 - 2n\pi)] + \sigma_v^2 \]
   (expressed in terms of normalised frequency \( \Omega = \omega T \)).

4.
5.
6.
7.
8. 11.2 ms
9.
10. (a)

\[ S_X(e^{j\Omega}) = \frac{\pi A^2}{2} \sum_{n=-\infty}^{+\infty} \left[ \delta(\Omega - \omega_1 T - 2n\pi) + \delta(\Omega + \omega_1 T - 2n\pi) \right] \]

\[ + \frac{\pi B^2}{2} \sum_{n=-\infty}^{+\infty} \left[ \delta(\Omega - \omega_2 T - 2n\pi) + \delta(\Omega + \omega_2 T - 2n\pi) \right] + \sigma_v^2 \]

(expressed in terms of normalised frequency \( \Omega = \omega T \)).

(b) 356 samples

11.
12.
13.
14.
15. \( b_0 = 2.0, \ b_1 = 1.8 \) by either method.

16. \( b_0 = 2.3345, \ a_0 = 1.0, \ a_1 = -0.4972 \)

Suitable questions from past papers:
4F7 2006 - 3 (Frequency estimation), 4 (autoregression/correlogram)
4F7 2005 - 3 (Periodogram/MA model), 4 (ARMA models)
4F7 2004 - 3 (MA models), 4 (Windowing/periodogram)
4F7 2003 - 3 (Periodogram), 4 (AR models)
Old I7 questions:
I7 2002 - 1 (AR Model), 4 (DFT - a bit off syllabus, but a challenging question - try it!)
I7 2001 - 1 (ARMA/MA model), 3 (DFT/windowing)
I7 2000 - 2 (Periodogram), 4 (AR Models)
I7 1999 - 2 (MA model)
I7 1998 - 1 (Windowing), 3 (AR Models)
Worked solutions

1. (a)

\[ X(e^{j\omega T}) = \sum_{n=-\infty}^{+\infty} x_n e^{-jn\omega T} \]
\[ = \sum_{n=-\infty}^{+\infty} \exp(i\omega_0 n)e^{-in\omega T} \]
\[ = \sum_{n=-\infty}^{+\infty} \exp(in(\omega_0 - \omega T)) \]
\[ = 2\pi \sum_{m=-\infty}^{+\infty} \delta(\omega T + 2\pi m - \omega_0) \]

since \( \sum_{n=-\infty}^{+\infty} \exp(in(\omega_0 - \omega T)) \) is the Fourier series representation of a periodic train of impulses.

(b) Using \( \sin(\omega_0 n) = 1/2j(\exp(+i\omega_0 n) - \exp(-i\omega_0 n)) \)

we have, by superposition:

\[ X(e^{j\omega T}) = \frac{1}{j\pi} \sum_{m=-\infty}^{+\infty} \delta(\omega T + 2\pi m - \omega_0) - \delta(\omega T + 2\pi m + \omega_0) \]

2. (a)

\[ X(e^{j\omega T}) = \sum_{n=-\infty}^{+\infty} x_n e^{-jn\omega T} \]
\[ = \sum_{n=0}^{N-1} \exp(i\omega_0 n)e^{-in\omega T} \]
\[ = \sum_{n=0}^{N-1} \exp(in(\omega_0 - \omega T)) \]
\[ = \frac{1 - \exp(i(\omega_0 - \omega T))^N}{1 - \exp(i(\omega_0 - \omega T))} \]
\[ = \exp(i(N - 1)(\omega_0 - \omega T)/2) \frac{\sin((\omega_0 - \omega T)N/2)}{\sin((\omega_0 - \omega T)/2)} \]

and

\[ |X(e^{j\omega T})| = \left| \frac{\sin((\omega_0 - \omega T)N/2)}{\sin((\omega_0 - \omega T)/2)} \right| \]

with \( \omega_0 = 0.2\pi \), see figure.

(b) Get this by superposition of two complex exponentials as in part a). See figure.
Note, zeros occur whenever $(\omega_0 - \omega T)N/2 = k\pi$, $k = 1, 2, 3, ... -1, -2, -3, ...$

Central lobe width $= 4\pi / N$

Figure 1: plot of $|X(e^{i\omega T})| = \left\lvert \frac{\sin((\omega_0 - \omega T)N/2)}{\sin((\omega_0 - \omega T)/2)} \right\rvert$

Figure 2: plot of $|X(e^{i\omega T})|$ for $\sin(\omega_0 n)$
3. (a) Standard material - see e.g. 3F3 lecture notes. (b) Get this by noting that the sine and noise terms are uncorrelated. Hence you can calculate the power spectrum of each term and add them together to get the result.

4. (a)

\[ x(n) = \frac{1}{N} \sum_{p=0}^{N-1} X(p) e^{j \frac{2\pi}{N} np} \]

\[ \therefore x(-n) = \frac{1}{N} \sum_{p=0}^{N-1} X(p) e^{-j \frac{2\pi}{N} np} \]

\[ = \frac{1}{N} \sum_{p=0}^{N-1} X(p) e^{j \frac{2\pi}{N}(N-n)p} \]

\[ \therefore x(-n) = x(N - n) \]

Similarly \( x(n + kN) = x(n) \quad k = \text{integer} \).

Thus the data are periodic. One way of viewing this is that the DFT is derived by sampling the Discrete Time Fourier Transform (DTFT) at a set of uniformly spaced frequencies. Sampling in the frequency domain leads to periodic repetition of time domain data in just the same way as sampling in the time domain leads to periodic repetition of spectra.

(b)

\[ X(p) = \sum_{n=0}^{N-1} x(n) e^{-j \frac{2\pi}{N} np} \]

\[ \therefore X(-p) = \sum_{n=0}^{N-1} x(n) e^{+j \frac{2\pi}{N} np} \]

\[ = \sum_{n=0}^{N-1} x(n) e^{-j \frac{2\pi}{N} n(N-p)} \]

\[ X(-p) = X(N - p) \]

Similarly \( X(p + kN) = X(p) \quad k = \text{integer} \).

(c)

Let:

\[ x(n - q) \leftrightarrow X'(p) \]

\[ X'(p) = \sum_{n=0}^{N-1} x(n - q) e^{-j \frac{2\pi}{N} np} \]

Let \( m = n - q \) then:

\[ X'(p) = \sum_{m=-q}^{N-1-q} x(m) e^{-j \frac{2\pi}{N}(m+q)p} \]
\[ e^{-j\frac{2\pi}{N} q p} \sum_{m=-q}^{N-1-q} x(m) e^{-j\frac{2\pi}{N} m p} \]

Both \( x(m) \) and \( e^{-j\frac{2\pi}{N} m p} \) are periodic in \( m \) with period \( N \) so that the summation can be carried out over \( m = 0, 1, \ldots, N - 1 \) which gives the desired result; this observation can be justified more formally if desired.

A more straightforward method is to approach the problem from the other end. Let:

\[ x'(n) \leftrightarrow e^{-j\frac{2\pi}{N} q p} X(p) \]

\[ x'(n) = \frac{1}{N} \sum_{p=0}^{N-1} [e^{-j\frac{2\pi}{N} q p} X(p)] e^{j\frac{2\pi}{N} n p} \]

\[ = \frac{1}{N} \sum_{p=0}^{N-1} X(p) e^{j\frac{2\pi}{N} (n-q) p} \]

\[ = x(n-q) \]

(d) Let \( x'(n) \leftrightarrow X_1(p) X_2(p) \). Then:

\[ x'(n) = \frac{1}{N} \sum_{p=0}^{N-1} X_1(p) X_2(p) e^{j\frac{2\pi}{N} n p} \]

Substitute

\[ X_1(p) = \sum_{q=0}^{N-1} x_1(q) e^{-j\frac{2\pi}{N} q p} \]

\[ x'(n) = \frac{1}{N} \sum_{p=0}^{N-1} \left\{ \sum_{q=0}^{N-1} x_1(q) e^{-j\frac{2\pi}{N} q p} \right\} X_2(p) e^{j\frac{2\pi}{N} n p} \]

\[ = \sum_{q=0}^{N-1} x_1(q) \frac{1}{N} \sum_{p=0}^{N-1} X_2(p) e^{j\frac{2\pi}{N} (n-q) p} \]

\[ = \sum_{q=0}^{N-1} x_1(q) x_2(n-q) \]

This is called a circular convolution in that \( x_1(n) \) and \( x_2(n) \) are periodic.

5.

\[ X(p) = \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N} n k} e^{-j\frac{2\pi}{N} n p} \]

\[ = \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N} n (k-p)} \]
This is a geometric progression and may be summed as:

\[ X(p) = \frac{1 - e^{j2\pi(k-p)}}{1 - e^{j\frac{2\pi}{N}(k-p)}} \]

\[ \therefore |X(p)| = \left| \frac{\sin \pi(k-p)}{\sin \frac{\pi}{N}(k-p)} \right| \]

From this expression and from figure 3 it can be seen that if \( k \) is an integer then the spectrum has a single component at frequency \( p = k \). If, however, \( k \) is not an integer then the spectrum is smeared. One way of looking at this is that the DFT is effectively the Fourier series of the periodic repetition of the \( N \) data points. Now if \( k \) is an integer, the periodic repetition will be continuous. If \( k \) is not an integer then the periodic repetition is discontinuous and the Fourier series will contain many frequency components.

![Time Domain](k=3.0)

![Frequency Domain](k=3.0)

![Time Domain](k=3.5)

![Frequency Domain](k=3.5)

Figure 3:

6.

\[ X(p) = \sum_{n=0}^{N-1} [x_1(n) + j x_2(n)] e^{-j \frac{2\pi}{N} np} \]

\[ X(N-p) = \sum_{n=0}^{N-1} [x_1(n) + j x_2(n)] e^{-j \frac{2\pi}{N} (N-p)} \]

\[ \sum_{n=0}^{N-1} [x_1(n) + j x_2(n)] e^{+j \frac{2\pi}{N} np} \]
\[ \therefore X^*(N - p) = \sum_{n=0}^{N-1} [x_1(n) - j x_2(n)] e^{-j \frac{2\pi}{N} np} \]

\[ \therefore X(p) + X^*(N - p) = 2 \sum_{n=0}^{N-1} x_1(n) e^{-j \frac{2\pi}{N} np} = 2 X_1(p) \]

and \[ X(p) - X^*(N - p) = 2j \sum_{n=0}^{N-1} x_2(n) e^{-j \frac{2\pi}{N} np} = 2j X_2(p) \]

7. The solution really amounts to doing the first stage of the FFT algorithm to convert the DFT of \(2N\) data points into 2 DFTs of \(N\) data points and using the result from question 6 to evaluate the 2 DFTs.

\[ X(p) = \sum_{n=0}^{2N-1} x(n) e^{-j \frac{2\pi}{2N} np} \quad p \in \{0, 2N - 1\} \]

\[ = \sum_{n=0}^{N-1} x(2n) e^{-j \frac{2\pi}{N} 2np} + \sum_{n=0}^{N-1} x(2n + 1) e^{-j \frac{2\pi}{N} (2n+1)p} \]

\[ = X_1(p) + e^{-j \frac{2\pi}{N} p} X_2(p) \quad (1) \]

where:

\[ X_1(p) = \sum_{n=0}^{N-1} x(2n) e^{-j \frac{2\pi}{N} np} \quad X_2(p) = \sum_{n=0}^{N-1} x(2n + 1) e^{-j \frac{2\pi}{N} np} \quad p \in \{0, N - 1\} \]

and

\[ X(p + N) = X_1(p) - e^{-j \frac{2\pi}{N} p} X_2(p) \]

Direct evaluation of the \(2N\) point DFT requires \(\frac{2N}{2} \log_2(2N)\) operations whereas evaluation of equation 1 requires:

\[ \left[ \frac{N}{2} \log_2(N) + \frac{N}{2} \log_2(N) + N \right] \]

\[ = N \left[ \log_2(N) + 1 \right] \]

\[ = N \left[ \log_2(N) + \log_2(2) \right] \]

\[ = N \log_2(2N) \]

However \(X_1(p)\) and \(X_2(p)\) are the DFTs of real data sequences so can be evaluated as in question 6 thus giving an approximate halving of the computation for large \(N\). The cost of one complex DFT of length \(N\) is \(\frac{N}{2} \log_2 N\) plus cost of the multiplication in (1) which is \(N\). Divide this total cost by \(N \log_2(2N)\) gives 0.5 as \(N \to \infty\)
8. The first part is standard bookwork

The spectrum of the rectangle window is:

\[ W(\omega) = \int_{-T_w/2}^{+T_w/2} \exp(-j\omega t) dt \]

\[ = T_w \text{sinc}(\omega T_w/2) \]

[Note the window can be centered on any time value; \( t = 0 \) is a convenient choice]

The 3dB point is

\[ |\text{sinc}(\omega T_w/2)| = 1/\sqrt{2} \]

i.e.

\[ \omega T_w/2 \approx 0.44 \]

(find this by trial and error or by plotting in Matlab)

We require that the two central lobes do not overlap. Hence the gap between the two frequencies must be greater than \( 2 \times 0.44/T_w \).

Now \( \omega_2 - \omega_1 = 2\pi \times 0.1 \times 10^3 \text{rad.s}^{-1} \). Hence

\[ 2 \times 0.88/T_w = 2\pi \times 0.1 \times 10^3 \]

i.e. \( T_w = 11.2 \text{ms} \). Note that this is a highly approximate estimate - a longer window length would be needed to ensure resolvability in all cases. The complex sidelobe structure means that the resolvability depends on the precise value of \( \omega_2 - \omega_1 \). Also, any noise in measurements can have an adverse effect on resolvability.

Comment: the extra factor of 2 in \( 2 \times 0.88/T_w \) arises so as to make sure the 3dB point of the \( \omega_1 \) component exactly matches the 3dB point of the \( \omega_2 \) component.

9. Let the signal be \( g(n), n \in \{0, M - 1\} \) and this is padded with zeros to give:

\[ g_{\text{pad}} = \begin{cases} 
  g(n) & n = 0, 1, \ldots, M - 1 \\
  0 & n = M, M + 1, \ldots, N - 1 
\end{cases} \]

Now the DTFT of the unpadded sequence is given by:

\[ G(e^{j\omega T}) = \sum_{n=0}^{M-1} g(n) e^{-jnm\omega T} \]

(Note that we are implicitly assuming that \( g(n) \) is zero outside the interval \( n = 0, 1, \ldots, M - 1 \)).

Sampling the DTFT at \( \omega T = \frac{2\pi}{N} p \) gives the normal \( M \) point DFT. However if the DTFT is sampled at frequencies \( \omega T = \frac{2\pi}{N} p \) then:

\[ G(e^{j\frac{2\pi}{N} p}) = \sum_{n=0}^{M-1} g(n) e^{-j\frac{2\pi}{N} np} \]

\[ = \sum_{n=0}^{N-1} g_{\text{pad}}(n) e^{-j\frac{2\pi}{N} np} \]
which is the DFT of the padded sequence so we see that the N-point DFT of the
padded sequence and the M-point DFT of the unpadded sequence correspond
to sampling the DTFT at N frequency points and M frequency points respectively.
Thus the padded DFT gives no greater frequency resolution (say between 2 closely
spaced sinusoids) but simply evaluates the spectrum at more frequencies; the effects
of leakage and smearing will be the same.

10. (a) From question 3 we have the power spectrum of a single random phase sine
wave in noise. To get the two-sine version, notice that both sine terms and
the noise term are mutually uncorrelated (check this if you are unsure). Hence
to get overall power spectrum, just add together the power spectra of the sine
waves with that of the noise (white).

(b) Expected value of the periodogram is (see lecture notes):

\[
E[\hat{S}_X(e^{j\omega T})] = \frac{1}{2\pi} \int_{-\pi}^{\pi} W(\nu T)S_X(e^{j(\omega-\nu)T}) \, d\nu T
\]

(2)
i.e. the convolution of the true power spectrum with the Bartlett window
(assuming biased form for the autocorrelation function estimate). The convo-
lution is easy to sketch since the power spectrum is a train of delta functions
plus a noise floor.

(c) (Assume T = 1.) 6dB bandwidth for Bartlett window is $1.78 \times 2\pi / N$, where N
is the window length. Hence, frequency resolution is approximately $2 \times 1.78 \times
2\pi / (2N) = 0.01\pi$ [note Bartlett window is of length 2N-1 for the periodogram
estimate]. Hence $N \approx 356$.

11. (a)

\[
E[\hat{S}_M(e^{j\omega T})] = \frac{1}{NU} E\left[\sum_{n=0}^{N-1} w_n x_n e^{-j\omega T}\right]^2
\]

\[
= \frac{1}{NU} E\left[\sum_{n=-\infty}^{\infty} w_n x_n e^{-j\omega T} \sum_{m=-\infty}^{\infty} w_m x_m e^{+j\omega T}\right]
\]

\[
= \frac{1}{NU} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} w_n w_m E[x_n x_m] e^{-j(n-m)\omega T}
\]

\[
= \frac{1}{NU} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} w_n w_m R_{XX}[n-m] e^{-j(n-m)\omega T}
\]

\[
= \frac{1}{NU} \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} w_n w_{n-k} R_{XX}[k] e^{-jk\omega T}
\]

with $n - m = k$

\[
= \frac{1}{NU} \sum_{k=-\infty}^{\infty} \left\{ \sum_{n=-\infty}^{\infty} w_n w_{n-k} \right\} R_{XX}[k] e^{-jk\omega T}
\]

\[
= \frac{1}{NU} \sum_{k=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} w_{mk} R_{XX}[k] e^{-jk\omega T}
\]
where
\[ w_{Mk} = \sum_{n=-\infty}^{\infty} w_n w_{n-k} = w_k * w_{-k} \]
as required.

Hence, by Fourier transform convolution theorem:
\[ E[\hat{S}_M(e^{j\omega_T})] = \frac{1}{2\pi NU} S_X(e^{j\omega_T}) * |W(e^{j\omega_T})|^2 \]

(b) Modified periodogram allows choice of a window function with suitable spectral leakage and spectral smearing properties to the application. This contrasts with the periodogram, in which the windowing function is fixed as the rectangular window - narrow central lobe but very severe sidelobes.

12. (a) Power spectrum is a train of delta functions centred at frequency \( \omega_0 \):
\[ S(e^{i\Omega}) = 2\pi \sum_{n=-\infty}^{+\infty} \delta(\Omega - \omega_0 T + 2\pi n) \]

(b) Assuming the biased autocorrelation estimator, we have the periodogram as:
\[ N \times \hat{S}(e^{i\Omega}) = \left| \sum_{n=0}^{N-1} x_n \exp(-in\Omega) \right|^2 \]
\[ = \left| \sum_{n=0}^{N-1} \exp(i(n\omega_0 T + \phi)) \exp(-in\Omega) \right|^2 \]
\[ = \left| \exp(i\phi) \sum_{n=0}^{N-1} \exp(i(n\omega_0 T)) \exp(-in\Omega) \right|^2 \]
\[ = \left| \sum_{n=0}^{N-1} \exp(i(n\omega_0 T)) \exp(-in\Omega) \right|^2 \]
\[ = \left| \sum_{n=0}^{N-1} \exp(-in(\Omega - \omega_0 T)) \right|^2 \]

[This can be simplified further, but unnecessary for this question] The important point here for the next parts is that the periodogram estimate does not depend on the value of the random variable \( \phi \). Hence the variance of the periodogram estimate is zero, see next part.

(c) The mean is the rectangular window spectrum shifted across in frequency to center frequency \( \omega_0 \). The variance is, however, zero. Thus in this case the variance is not the rule of thumb. i.e. for single complex exponentials the periodogram gives no variability. This is because the periodogram of a complex exponential is constant whatever the phase of the exponential

13. (a) Variance of periodogram is approximately \( \sigma^4 \) for all data lengths, becoming exact as data length goes to infinity. This means that the variance does not decrease as data length increases and therefore periodogram will have unacceptable variability for many noise like processes.
(b) Each of the $K$ subsequences of data are statistically independent so that the periodogram estimates for each subsequence are also statistically independent. Consider a particular frequency component $\hat{S}_X^{(k)}(e^{j\omega_i T})$ from each of the $K$ periodograms. These $K$ components may be regarded as statistically independent random numbers with variance $\sigma^2$. In order to ease the notation, let:

$$Z_k \equiv \hat{S}_X^{(k)}(e^{j\omega_i T})$$

The variance $\alpha^2$ of the Bartlett estimate is:

$$\alpha^2 = E\left\{ \frac{1}{K} \sum_{k=0}^{K-1} Z_k - \frac{1}{K} \sum_{k=0}^{K-1} E\{Z_k\}\right\}^2$$

$$= \frac{1}{K^2} E\left\{ \sum_{k=0}^{K-1} Z_k \right\}^2 - 2\bar{Z}^2 + \bar{Z}^2$$

where:

$$\bar{Z} = E\{Z_k\}$$

$$\therefore \alpha^2 = \frac{1}{K^2} E\left\{ \sum_{k=0}^{K-1} \sum_{j=0}^{K-1} Z_k Z_j \right\} - \bar{Z}^2$$

Now remembering that $Z_k$ is a random variable with mean value $\bar{Z}$ we can write:

$$E\{Z_k Z_j\} = \begin{cases} \sigma^4 + \bar{Z}^2 & k = j \\ \bar{Z}^2 & k \neq j \end{cases}$$

$$\therefore \alpha^2 = \frac{1}{K^2} \sum_{k} (\sigma^4 + \bar{Z}^2) + \frac{1}{K^2} (K^2 - K) \bar{Z}^2 - \bar{Z}^2$$

$$\therefore \alpha^2 = \frac{\sigma^4}{K}$$

i.e. the variance of the spectrum estimate has been reduced by a factor of $K$. To be more precise,

$$\text{var}(Z_k) = \sigma^4 \left( 1 + \left\{ \frac{\sin(N_B \omega_i T)}{N_B \sin(\omega_i T)} \right\}^2 \right)$$

and not just $\sigma^4$. When the data is not segmented, the variance has the same expression except that $N_B$ should be replaced by $N_B K$. The ratio of these two quantities tends to $1/K$ as $N = N_B K$ tends to infinity with $K$ fixed.

(c)

$$E[\hat{S}_X(e^{j\omega T})] = E\left[ \frac{1}{K} \sum_{k=0}^{K-1} \hat{S}_X^{(k)}(e^{j\omega T}) \right]$$

$$= \frac{1}{K} \sum_{k=0}^{K-1} E[\hat{S}_X^{(k)}(e^{j\omega T})]$$

Now each expectation term is an expectation of a periodogram estimate for each sub-block. Hence, as for the periodogram the method is biased but asymptotically unbiased.
(d) Clearly each periodogram in the summation corresponds to a data window length \( N_B = N/K \), where \( N \) is the total length. (Note the value of the window length in the expected value of the periodogram is \( 2 \times \text{datalength} - 1 \)) Hence each periodogram estimate in the Bartlett summation has \( K \) times poorer resolution, since central lobe of window spectrum is \( K \) times wider than that of the full periodogram estimate for all \( N \) data points.

14. If impulse response of the ARMA filter is \( h_n \), then we know from standard linear systems theory that the filter is stable if and only if

\[
\sum_{n=0}^{\infty} |h_n| < \infty
\]

Now ARMA signal can be written as the output of a linear system with white noise input \( \{w_n\} \):

\[
x_n = \sum_{i=0}^{\infty} h_i w_{n-i}
\]

Checking for WSS:

First the mean

\[
\mu_X[n] = E[x_n] = E[\sum_{i=0}^{\infty} h_i w_{n-i}] = \sum_{i=0}^{\infty} h_i E[w_{n-i}] = \sum_{i=0}^{\infty} h_i \mu_w
\]

i.e. mean is constant since mean of stationary input process is constant \( \mu_w \).

Then the autocorrelation function:

\[
R_{XX}[n, m] = E[x_n x_m] = E[\sum_{i=0}^{\infty} h_i w_{n-i} \sum_{j=0}^{\infty} h_j w_{m-j}]
\]

\[
= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} h_i h_j E[w_{n-i} w_{m-j}]
\]

\[
= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} h_i h_j R_{WW}[(n - m) - (i - j)]
\]

since \( \{w_n\} \) is stationary. Hence autocorrelation function depends only on time difference \( n - m \), as required for WSS. However, we should verify that the sum that defines \( R_{XX}[n, m] \) exists. It does provided \( R_{XX}[n, m] \) is absolutely summable. \( R_{XX}[n, m] \) is absolutely summable since

\[
\sum_{i,j} |h_i| |h_j| < \infty
\]

This will not be the case for an unstable process.

Finally the variance of the process is finite since \( \sigma_X^2 = R_{XX}[0] - \mu_X^2 \).

Derivation of the autocorrelation function for the ARMA model was given in the lecture notes.
15. Either method should give:

\[ b_0 = 2.0 \quad b_1 = 1.8 \]

Remember that the spectral factorisation method only gives the roots of the polynomial and the scaling must be calculated separately.

By direct solution \((Q = 1)\):

\[
\begin{bmatrix}
R_{XX}[0] \\
R_{XX}[1]
\end{bmatrix} = \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} \sum_{q=0}^1 b_q^2 \\ \sum_{q=1}^2 b_q b_{q-1} \end{bmatrix} = \begin{bmatrix} b_0^2 + b_1^2 \\ b_1 b_0 \end{bmatrix}.
\]

Inserting values:

\[
\begin{bmatrix} 7.24 \\ 3.6 \end{bmatrix} = \begin{bmatrix} b_0^2 + b_1^2 \\ b_1 b_0 \end{bmatrix}
\]

Now solve to get:

\[
b_1 = 3.6/b_0 \\
7.24 = b_0^2 + 3.6^2/b_0^2 \\
b_0^4 - 7.24b_0^2 + 3.6^2 = 0
\]

\((b_0^2 - 4)(b_0^2 - 3.24) = 0.\) For \(b_0 = \pm 2,\) \(b_1 = \pm 1.8\)

By spectral factorization: first solve for the zeros of \(\sum_{r=-Q}^Q R_{XX}[r]z^{-r}.\)

\[
R_{XX}[-1]z + R_{XX}[0] + R_{XX}[1]z^{-1} = 3.6z + 7.24 + 3.6z^{-1} = z \left(3.6 + 7.24z^{-1} + 3.6z^{-2}\right) = z \left(z^{-1} + \frac{7.24 + 0.76}{7.2}\right) \left(z^{-1} + \frac{7.24 - 0.76}{7.2}\right) = \left(1 + \frac{8}{7.2}z\right) \left(z^{-1} + \frac{6.48}{7.2}\right)
\]

The root in the unit circle is \(-7.2/8 = -0.9.\)

Now \(g \left(1 - z^{-1}(-0.9)\right)\) is to be solved for \(g.\)

\[
c_0 = g^2 + 0.9^2g^2 \\
7.24 = g^2(1 + 0.9^2) \\
g = \sqrt{\frac{7.24}{1 + 0.9^2}} = 2
\]

So the MA model is \(B(z) = 2 + 1.8z^{-1},\) or \(b_0 = 2, b_1 = 1.8.\) The MA spectrum corresponding to the model is shown in figure 4
16. The AR coefficients are:

\[ b_0 = 2.3345 \quad a_0 = 1 \quad a_1 = -0.4972 \]

To fit the AR model use the Yule-Walker equations \( P=1, Q=0 \):

\[
R_{XX}[0]a_1 = R_{XX}[-1] \\
\begin{align*}
0 = R_{XX}[0] - 3.6724 \\
a_1 = -3.6724.
\end{align*}
\]

Now solve for \( b_0 \):

\[
b_0^2 = [ R_{XX}[0] \quad R_{XX}[-1] ] \begin{bmatrix} 1 & a_1 \end{bmatrix}^T \\
= R_{XX}[0] - \frac{R_{XX}[1]^2}{R_{XX}[0]} = 2.335^2
\]

\[
H(z) = \frac{b_0}{1 + a_1 z^{-1}} = \frac{2.335}{1 - \frac{3.6724}{7.24} z^{-1}}.
\]

Power spectrum is \( S(e^{j\omega T}) = |H(e^{j\omega T})|^2 \). The AR spectrum corresponding to the model is shown in figure 5.

Without any prior knowledge of the physical system which produced the signals, one spectral estimate should not be preferred over the other. However, the 1st order MA model assumes that the signal correlation is zero for lags greater than 1.
Figure 5:

whereas the AR model assumes that the correlation function satisfies the AR difference equation so that the correlation function is not zero for lags greater than 1. It might be argued that this is a more reasonable reflection of what might be the case in the system which generated the signal.

S.S. Singh (≥ 2008), Simon Godsill, November 2003, Peter Rayner 1999