4F7 Adaptive filters and Spectrum estimation

Examples Paper 2

1. *Regularised LMS.* Let
$$J(\mathbf{h}) = E\{e^2(n)\} + \alpha \|\mathbf{h}\|^2$$
where
$$e(n) = d(n) - \mathbf{h}^{\mathrm{T}}\mathbf{u}(n).$$
   - Show that the LMS update rule for $\mathbf{h}(n)$ is
   $$\mathbf{h}(n+1) = (1-\mu\alpha)\,\mathbf{h}(n) + \mu\mathbf{u}(n)\,e(n).$$
   - Show that if $\lim_{n\to\infty} E\{\mathbf{h}(n)\}$ exists then it satisfies
   $$\overline{\mathbf{h}} = \lim_{n\to\infty} E\{\mathbf{h}(n)\} = (\mathbf{R}+\alpha\mathbf{I})^{-1}\mathbf{p}$$
   where $\mathbf{R} = E\left\{\mathbf{u}(n)\,\mathbf{u}^{\mathrm{T}}(n)\right\}$, $\mathbf{p} = E\{\mathbf{u}(n)\,d(n)\}$ and clearly state any approximations used.
   - What is the requirement for the stepsize μ to ensure convergence? When could the use of this algorithm be beneficial?

2. A constant variable C is measured through two different sensors. The measurements are noisy and have different accuracy,
$$\begin{aligned} y_1 &= C + e_1 \\ y_2 &= C + e_2 \end{aligned}$$
where $\mathbf{e} = (e_1\ e_2)^{\mathrm{T}}$ is a zero-mean noise term of covariance
$$\mathbf{R} = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix}.$$
Consider the following estimate of C
$$\widehat{C} = a_1 y_1 + a_2 y_2.$$
Find (a_1, a_2) so that $\widehat{C}$ is unbiased and has minimum variance.

3. Consider the following state-space model

$$\begin{aligned}\mathbf{x}(n) &= \mathbf{A}\mathbf{x}(n-1)+\mathbf{B}\mathbf{v}(n)\\ \mathbf{y}(n) &= \mathbf{C}\mathbf{x}(n)+\mathbf{w}(n)\end{aligned}$$

where $\{\mathbf{w}(n)\}$ is a white noise sequence but noise sequence $\{\mathbf{v}(n)\}$ satisfies

$$\mathbf{v}(n)=\mathbf{A}_v\mathbf{v}(n-1)+\mathbf{B}_v\mathbf{e}(n)$$

where sequence $\{\mathbf{e}(n)\}$ is a white noise sequence. How could the Kalman filter be applied to estimate $\mathbf{x}(n)$ from the observation sequence $\mathbf{y}(n)$?

4. Assume we observe for $n \geq 0$

$$y(n)=\alpha+w(n) \tag{1}$$

where $\{w(n)\}$ is a zero-mean white noise sequence of variance σ_w^2 and α is a random variable with mean zero and standard deviation σ_α.

- Give the state-space representation for the signal (1).
- Derive the Kalman filter to obtain the l.m.m.s.e. $\widehat{\alpha}(n)$ of α given $\{y(0),\ldots,y(n)\}$. What is the limit of the covariance of this estimate as $n\to\infty$?

5. Consider the following autoregressive-moving average model

$$\begin{aligned}\alpha(n) &= \sum_{i=1}^{p} a_i\alpha(n-i)+v(n),\\ \beta(n) &= \sum_{i=0}^{q-1} b_i\alpha(n-i)+w(n).\end{aligned}$$

and give a state-space representation of this model. Distinguish the cases where $p\geq q$ and $p<q$.