4F7 (Adaptive Filters and) Spectrum Estimation

Fitting the Moving Average Model

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Parametric methods: The MA Model \((P = 0)\)

- The MA model is a FIR filter driven by white noise. The Yule-Walker equations simplifies to

\[
\begin{bmatrix}
R_{XX}[0] \\
R_{XX}[1] \\
\vdots \\
R_{XX}[Q]
\end{bmatrix}
= 
\begin{bmatrix}
c_0 \\
c_1 \\
\vdots \\
c_Q
\end{bmatrix}
\]

\((1)\)

- However the solution of this equation is not trivial since the \(c_i\), given in a previous lecture, is the convolution of the MA coefficients \(b_i\) and the impulse response of the ARMA model

- The ARMA model impulse response is

\[
x_n = \sum_{m=-\infty}^{\infty} h_m w_{n-m} \quad \text{(causal)} = \sum_{m=0}^{\infty} h_m w_{n-m}
\]

- Comparing with the MA model, \(\sum_{m=0}^{Q} b_m w_{n-m}\), we see that \(h_i = b_i\)
• Using \( h_i = b_i \), the expression for \( c_r \) given before may be rewritten as:

\[
c_r = \begin{cases} 
\sum_{q=r}^{Q} b_q b_{q-r} & \text{if } r \leq Q \\
0 & \text{if } r > Q 
\end{cases}
\tag{2}
\]

• (2) is valid for negative \( r \) too and for \( r < 0 \)

\[c_r = c_{|r|}\]

• The convolution of the following two infinite sequences

\[\ldots 0, \ b_0, \ b_1, \ \ldots \ b_Q, \ 0, \ \ldots\]

\[\ldots 0, \ b_Q, \ \ldots \ b_1, \ b_0, \ 0, \ \ldots\]

gives \( c_r \), i.e. \( c_r = (\{b_{-n}\} * \{b_n\})(r)\)
Let $\mathcal{Z} (\{x_n\}) = \sum_{n=-\infty}^{+\infty} x_n z^{-n}$ be the ‘bilateral’ z-transform of the sequence $\{x_n\}$

$$
\sum_{r=-Q}^{Q} c_r z^{-r} = \mathcal{Z} (\{b_n\} * \{b_{-n}\})
$$

$$
= B(z)B(z^{-1}) \quad \text{ (since } \mathcal{Z}\{b_{-n}\} = B(z^{-1}))
$$

and substituting for $c_r$ from equation 1:

$$
B(z) B(z^{-1}) = \sum_{r=-Q}^{Q} RXX[|r|] z^{-r}
$$

- Let the zeros of $B(z)$, \{z \in \mathbb{C} : B(z) = 0\}, be $n_1, n_2, \ldots, n_Q$. Then, it is obvious that $n_1^{-1}, n_2^{-1}, \ldots, n_Q^{-1}$ are the zeros of $B(z^{-1})$.

- The zeros of $B(z)B(z^{-1})$ are $\{n_i, n_i^{-1}\}_{i=1}^{Q}$
  If a zero $n_i$ lies inside (or on) the unit circle, then the corresponding
inverse zero \(1/n_i\) lies outside (or on) the unit circle

- Technical condition: assume all the zeros of \(B(z)\) lie inside the unit circle so that the MA process is invertible. Invertible means you can express \(w_n\) using \(x_n\) and its past values.

- Now identify the zeros of \(B(z)\) by finding the zeros of the RHS of (5) that lie within the unit circle.
• Once we have the roots $n_i$ of $B(z)$ it is straightforward to reassemble $B(z)$ from the zeros, up to an unknown scale factor $g$

• You can always write $B(z) = b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_Q z^{-Q}$ as

$$B(z) = g \prod_{i=1}^{Q} (1 - z^{-1} n_i)$$

$$= g(b'_0 + b'_1 z^{-1} + b'_2 z^{-2} + \cdots + b'_Q z^{-Q})$$

with $b'_0 = 1$.

• Solve for the scale factor $g$

$$\sum_{n=0}^{Q} b_n^2 = c_0 \quad \text{and} \quad \underbrace{c_0 = R_{XX}[0]}_{\text{eqn (2)}}$$

• Hence

$$\sum_{i=0}^{Q} (gb'_i)^2 = R_{XX}[0]$$
from which:

\[ g = \sqrt{\frac{R_{XX}[0]}{\sum_{i=0}^{Q} (b'_i)^2}} \]

and finally:

\[ b_i = g \times b'_i = \sqrt{\frac{R_{XX}[0]}{\sum_{i=0}^{Q} (b'_i)^2}} b'_i \]
Example  It is required to fit an MA model to the correlation data:

\[
\begin{bmatrix}
R_{XX}[0] \\
R_{XX}[1] \\
R_{XX}[2]
\end{bmatrix} = \begin{bmatrix}
4.06 \\
-2.85 \\
.9
\end{bmatrix}
\]

Therefore
\[
\sum_{r=-Q}^{Q} R_{XX}(r) z^{-r} = 0.9 z^{-2} - 2.85 z^{-1} + 4.06 - 2.85 z + 0.9 z^2
\]

Factorisation of this polynomial gives the roots:

\[0.8333 \pm j 0.6455\]
\[0.75 \pm j 0.5808 \ (n_1, n_2 \text{ roots inside unit circle})\]

which are plotted in figure 1. The roots inside the unit circle are identified
with $B(z)$.

$$B(z) = g(1 - z^{-1}n_1)(1 - z^{-1}n_2)$$

$$= g(1 - z^{-1}(n_1 + n_2) + z^{-2}n_1n_2)$$

$$= g(1 - z^{-1}1.5 + z^{-2}0.9)$$

Thus

$$B(z) = \sqrt{\frac{4.06}{1 + 1.5^2 + 0.9^2}}(1 - 1.5 z^{-1} + 0.9 z^{-2})$$

$$= 1 - 1.5 z^{-1} + 0.9 z^{-2}$$

and the corresponding MA model is:

$$x_n = w_n - 1.5 w_{n-1} + 0.9 w_{n-2}$$

where $w_n$ is white noise with variance equal to 1.
Figure 1: Zeros of $B(z), \ B(z^{-1})$