4F7 (Adaptive Filters and) Spectrum Estimation

Properties of the Periodogram

Sumeetpal Singh
Email : sss40@eng.cam.ac.uk

Modification history:
$\leq$ 2008: S. Godsill, P. Rayner
Revised S.S. Singh

1 Properties of an Estimator

- To evaluate how good an estimator is, characterise its bias and variance
- An estimator $\hat{\theta}$ of a random quantity θ is unbiased if the expected value of the estimate equals the true value, i.e.

  $$E[\hat{\theta}] = \theta$$

  Otherwise the estimator is termed biased. Variability an estimator has around its mean value is (or variance)

  $$\text{var}(\hat{\theta}) = E[(\hat{\theta} - E[\hat{\theta}])^2]$$

- A good estimator will make some suitable trade-off between low bias and low variance.

Now, apply these ideas to the periodogram ...

2 Expected Value of the Periodogram

- The expected value of the periodogram is

$$E[\hat{S}_X(e^{j\omega})] = E\left[\sum_{k=-(N-1)}^{N-1} \hat{R}_{XX}[k]\, e^{-jk\omega}\right] = \sum_{k=-(N-1)}^{N-1} E[\hat{R}_{XX}[k]]\, e^{-jk\omega}, \tag{1}$$

which is the DTFT of the expected autocorrelation function estimate

- $E[\hat{R}_{XX}[k]]$ depends on whether we used the 'biased' or 'unbiased' forms for $\hat{R}_{XX}$

- Consider first the unbiased form:

$$\begin{aligned} E[\hat{R}_{XX}[k]] &= E\left[\frac{1}{N-k}\sum_{n=0}^{N-1-k} x_n\, x_{n+k}\right] \\ &= \frac{1}{N-k}\sum_{n=0}^{N-1-k} E[x_n x_{n+k}] \\ &= \frac{1}{N-k}\sum_{n=0}^{N-1-k} R_{XX}[k] \\ &= R_{XX}[k] \end{aligned} \tag{2}$$

- Repeat calculation for the biased version:

$$E[\hat{R}_{XX}[k]] = \frac{N-k}{N} R_{XX}[k], \quad 0 \leq k < N \tag{3}$$

- In summary, noting that $\hat{R}_{XX}[-k] = \hat{R}_{XX}[k]$,

$$E[\hat{R}_{XX}[k]] = w_k R_{XX}[k], \qquad k = -N+1, ..., N-1$$

where, for the unbiased estimate,

$$w_k = \begin{cases} 1, & |k| < N \\ 0, & \text{otherwise} \end{cases} \quad \text{(Rectangular window)}$$

and for the biased estimate,

$$w_k = \begin{cases} \frac{N-|k|}{N}, & |k| < N \\ 0, & \text{otherwise} \end{cases} \quad \text{(Bartlett (triangular) window)}$$

- Substituting into the expression for $E[\hat{S}_X(e^{j\omega})]$ we obtain:

$$\begin{aligned} E[\hat{S}_X(e^{j\omega})] &= \sum_{k=-(N-1)}^{N-1} E[\hat{R}_{XX}[k]]\, e^{-jk\omega} \\ &= \sum_{k=-(N-1)}^{N-1} w_k R_{XX}[k]\, e^{-jk\omega} \\ &= \sum_{k=-\infty}^{\infty} w_k R_{XX}[k]\, e^{-jk\omega} \end{aligned}$$

- The DTFT of a *product* of two functions w_k and R_{XX} is equal to the *convolution* of their individual DTFTs:

$$E[\hat{S}_X(e^{j\omega})] = \frac{1}{2\pi}\int_{-\pi}^{\pi} W(e^{j\theta}) S_X(e^{j(\omega-\theta)})\, d\theta \tag{4}$$

where $S_X(.)$ is the true power spectrum and $W(.)$ is the DTFT of the particular window function w_k.

Consider now the biased and unbiased cases:

1. Biased. $W(.)$ is the DTFT of the Bartlett or triangular window:

$$W(e^{j\omega}) = \frac{1}{N}\left[\frac{\sin(N\frac{\omega}{2})}{\sin(\frac{\omega}{2})}\right]^2 \longrightarrow 2\pi\delta(\omega)$$

   ($2\pi\delta$ because $\int_{-\pi}^{\pi} W(e^{j\theta})d\theta = 2\pi$)

2. Unbiased. $W(.)$ is the DTFT of the rectangular window:

$$W(e^{j\omega}) = \left[\frac{\sin(2N-1)\frac{\omega}{2}}{\sin(\frac{\omega}{2})}\right]$$

- Note that the Bartlett window spectrum is always positive - hence the spectrum estimate is also positive.
- Rectangular window spectrum has negative parts, hence spectrum estimate can be negative (i.e. invalid estimate): a reason to prefer Bartlett
- Note also that both estimators are biased in that the expected value does not equal the true spectrum $S_X(e^{j\omega})$. However Bartlett asymptotically unbiased: $\lim_{N\to\infty} E[\hat{S}_X(e^{j\omega})] = S_X(e^{j\omega})$

3 Example: periodogram of white noise

- For a white noise process

$$R_{XX}[k] = \begin{cases} \sigma^2 & k = 0 \\ 0 & \text{otherwise} \end{cases} = \sigma^2 \delta_k$$

where δ_k is the Kronecker delta-function.

- Substituting this into the expression for expected value of the periodogram:

$$\begin{aligned} E[\hat{S}_X(e^{j\omega})] &= \sum_{k=-(\infty)}^{\infty} w_k R_{XX}[k]\, e^{-jk\omega} \\ &= \sum_{k=-\infty}^{\infty} w_k \sigma^2 \delta_k\, e^{-jk\omega} = w_0 \sigma^2 \\ &= \sigma^2 \quad \text{(for both Bartlett \& rect. windows)} \end{aligned}$$

- Hence the periodogram is unbiased for white noise

4 Variance of the Periodogram

- The good news was that the periodogram is asymptotically unbiased:

$$\lim_{N\to\infty} E[\hat{S}_X(e^{j\omega})] = S_X(e^{j\omega})$$

- We would wish that it is also consistent. A consistent estimator is one which is asymptotically unbiased and whose variance tends to zero as $N \to \infty$.

- The variance of the periodogram cannot easily be analysed for general random processes. However, for a Gaussian random process it can be shown that:

$$\begin{aligned} \mathrm{var}(\hat{S}_X(e^{j\omega})) &= E[(\hat{S}_X(e^{j\omega}) - E[\hat{S}_X(e^{j\omega})])^2] \\ &\approx S_X(e^{j\omega})^2 \end{aligned}$$

  this result being exact for *white* Gaussian processes.

- Since this does not depend on N, the variance does not reduce to zero as N increases

5 Variance of periodogram - Gaussian white noise case

- It is generally harder to work out the variance of the periodogram for general processes. However, for white Gaussian noise it is straightforward but tedious
- To find the variance of the periodogram for white Gaussian noise , expand the formula directly

$$\text{var}\left(\hat{S}_X(e^{j\omega})^2\right) = E[\hat{S}_X(e^{j\omega})^2] - \left(E[\hat{S}_X(e^{j\omega})]\right)^2$$

- Expand the first term

$$E[\hat{S}_X(e^{j\omega})^2]$$

$$= E\left[\left(\frac{1}{N}\left|\sum_{n=0}^{N-1} x_n e^{-jn\omega}\right|^2\right)^2\right]$$

$$= \frac{1}{N^2}E[\sum_{n_1=0}^{N-1} x_{n_1}e^{-jn_1\omega}\sum_{n_2=0}^{N-1} x_{n_2}e^{+jn_2\omega}$$

$$\times \sum_{n_3=0}^{N-1} x_{n_3}e^{-jn_3\omega}\sum_{n_4=0}^{N-1} x_{n_4}e^{+jn_4\omega}]$$

$$= \frac{1}{N^2}\sum_{n_1=0}^{N-1}\sum_{n_2=0}^{N-1}\sum_{n_3=0}^{N-1}\sum_{n_4=0}^{N-1} E[x_{n_1}x_{n_2}x_{n_3}x_{n_4}]e^{-j(n_1+n_3-n_2-n_4)\omega}$$

- Possible values for $E[x_{n_1}x_{n_2}x_{n_3}x_{n_4}]$ are

  $3\sigma^4$ when $n_1 = n_2 = n_3 = n_4$,

  σ^2 when pairs of indices match, e.g. $n_1 = n_2, n_3 = n_4$,

  0 otherwise

- When $n_1 = n_2 = n_3 = n_4$,

$$E[x_{n_1}x_{n_2}x_{n_3}x_{n_4}] = E[x_n^4]$$
$$= \int_{-\infty}^{+\infty} x_n^4 \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x_n^2}{2\sigma^2}} dx_n = 3\sigma^4$$

  There are N such cases in the multiple summation, corresponding to $n_1 = 0, 1, 2, \ldots, N-1$.

- When 3 time indices are equal, e.g. $n_1 = n_2 = n_3$ and $n_4 \neq n_1$,

$$E[x_{n_1}x_{n_2}x_{n_3}x_{n_4}] = E[x_{n_1}^3 x_{n_4}] = E[x_{n_1}^3]E[x_{n_4}] = 0$$

  since x_{n_1} and x_{n_4} are independent

- When 2 pairs of time indices are equal, e.g. $n_1 = n_2$ and $n_3 = n_4$ and $n_1 \neq n_3$. Here,

$$E[x_{n_1} x_{n_2} x_{n_3} x_{n_4}] = E[x_{n_1}^2 x_{n_3}^2] = E[x_{n_1}^2] E[x_{n_3}^2] = \sigma^4.$$

- There are a number of ways for two pairs to be equal in the multiple summation:

  a) $n_1 = n_2$ and $n_3 = n_4$. There are $N \times (N-1)$ ways for this to happen:

$$\underbrace{\sum_{\substack{n_1=0 \\ n_2=n_1}}^{N-1}}_{N \text{ terms}} \quad \underbrace{\sum_{\substack{n_3=0 \\ n_3 \neq n_1 \\ n_4=n_3}}^{N-1}}_{N-1 \text{ terms}}$$

  and the exp. term in this case becomes $e^{j0} = 1$.

  b) $n_1 = n_3$ and $n_2 = n_4$. There are $N \times (N-1)$ ways for this to happen, and the exponential term in this case becomes $e^{-j(2n_1 - 2n_2)\omega}$.

  c) $n_1 = n_4$ and $n_2 = n_3$. There are $N \times (N-1)$ ways for this to happen, and the exponential term in this case becomes $e^{j0} = 1$.

- Try to verify that for all other cases $E[x_{n_1} x_{n_2} x_{n_3} x_{n_4}] = 0$

• Combining these together we have:

$$\frac{1}{N^2}\sum_{n_1=0}^{N-1}\sum_{n_2=0}^{N-1}\sum_{n_3=0}^{N-1}\sum_{n_4=0}^{N-1} E[x_{n_1}x_{n_2}x_{n_3}x_{n_4}]e^{-j(n_1+n_3-n_2-n_4)\omega}$$

$$= \frac{1}{N^2}\Bigg(3N\sigma^4 + N\times(N-1)\times\sigma^4 e^{j0} + \sum_{\substack{n_1 \\ n_1\neq n_2}}\sum_{n_2}\sigma^4 e^{-j(2n_1-2n_2)\omega} + N\times(N-1)\sigma^4\Bigg)$$

$$= \frac{1}{N^2}\left(3N\sigma^4 + N\times(N-1)\times\sigma^4 e^{j0}\right) + \frac{1}{N^2}\left(\sum_{n_1}\sum_{n_2}\sigma^4 e^{-j(2n_1-2n_2)\omega}\right) + \frac{1}{N^2}\left(-N\sigma^4 + N\times(N-1)\sigma^4\right)$$

$$= \frac{1}{N^2}\left(2N^2\sigma^4 + \sum_{n_1}\sum_{n_2}\sigma^4 e^{-j(2n_1-2n_2)\omega}\right)$$

For the transition from the second last line to the last line, use $\sum_{n=0}^{N-1} ar^n = a\frac{1-r^N}{1-r}$.

- Finally, looking at the second term in the variance formula:

$$\begin{aligned} &\text{var}(\hat{S}_X(e^{j\omega})) \\ &= E[\hat{S}_X(e^{j\omega})^2] - E[\hat{S}_X(e^{j\omega})]^2 \end{aligned}$$

this is simply the squared value of the expected value of the periodogram for white noise which we have calculated

$$E[\hat{S}_X(e^{j\omega})]^2 = (\sigma^2)^2 = \sigma^4$$

so that:

$$\begin{aligned} \text{var}(\hat{S}_X(e^{j\omega})) &= E[\hat{S}_X(e^{j\omega})^2] - E[\hat{S}_X(e^{j\omega})]^2 \\ &= \sigma^4 \left(1 + \left\{ \frac{\sin(N\omega)}{N\sin(\omega)} \right\}^2 \right) \\ &\approx \sigma^4 \quad \text{as } N \to \infty \\ &= S_X(e^{j\omega})^2 \end{aligned}$$

[See Matlab demo `periodogram_white_noise.m`]

6 Variance of periodogram - general case

- It is much more complex to evaluate the variance of the periodogram for a general random process. However, some approximations can be used to arrive at a similar expression for the Gaussian case.
- We can rewrite a stationary random process as a white noise process $v = \{v_n\}$ with power spectrum equal to σ^2 driving a linear filter $H(e^{j\omega})$:

$$v = \{v_n\} \longrightarrow \boxed{\; H(e^{j\omega}) \;} \longrightarrow x = \{x_n\}$$

- The power spectrum of such a process is:

$$S_X(e^{j\omega}) = \sigma^2 |H(e^{j\omega})|^2$$

- Now, define as usual windowed versions of $\{v_n\}$ and $\{x_n\}$:

$$v_{w,n} = \begin{cases} v_n, & n = 0, 1, ..., N-1 \\ 0, & \text{otherwise} \end{cases}$$

$$x_{w,n} = \begin{cases} x_n, & n = 0, 1, ..., N-1 \\ 0, & \text{otherwise} \end{cases}$$

- The windowed version $x_w = \{x_{w,n}\}$ is not equal to the convolution of $v_w = \{v_{w,n}\}$ with the filter $h = \{h_n\}$. However if the window is long compared to the length of the filter so that the transient effects are small then

  $$x_w \approx h * v_w$$

  and the corresponding approximate result when the DTFT is performed:

  $$X_w(e^{j\omega}) \approx V_w(e^{j\omega})H(e^{j\omega})$$

- To get the periodogram estimate:

  $$1/N|X_w(e^{j\omega})|^2 \approx 1/N|V_w(e^{j\omega})|^2|H(e^{j\omega})|^2$$

and hence:

$$
\begin{aligned}
&var(1/N|X_w(e^{j\omega})|^2) \\
&\approx var(1/N|V_w(e^{j\omega})|^2)(|H(e^{j\omega})|^2)^2 \\
&= \frac{1}{\sigma^4} var(1/N|V_w(e^{j\omega})|^2)(S_X(e^{j\omega}))^2
\end{aligned}
$$

- But, $v = \{v_n\}$ is white Gaussian noise, whose periodogram has variance equal to σ^4 when N is large. Hence:

$$\text{var}(\hat{S}_X(e^{j\omega}) = \text{var}(1/N|X_w(e^{j\omega})|^2) \approx S_X(e^{j\omega})^2$$

as required.

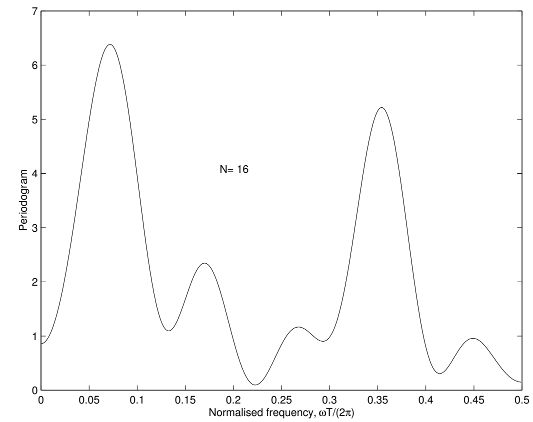
7 Example: Sine-wave plus Gaussian noise

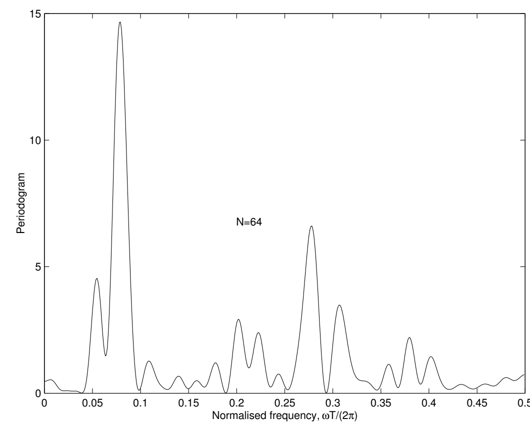
Consider a random process of the form:

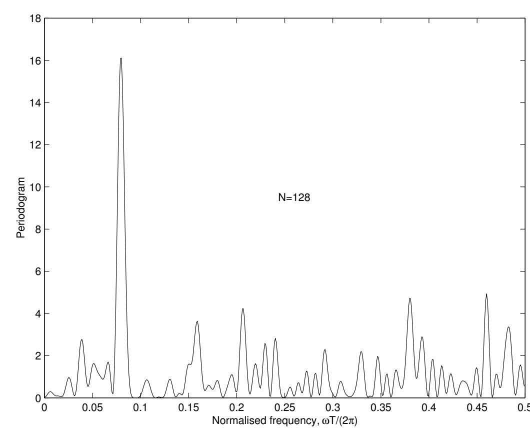
$$x_n = \sin(\omega nT + \phi) + v_n$$

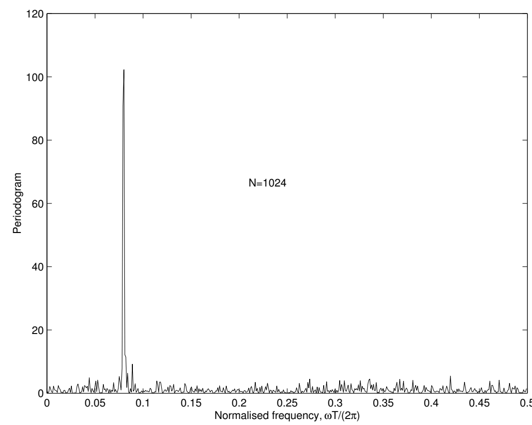
where $\{v_n\}$ is a white Gaussian noise process and ϕ is a random phase distributed uniformly between 0 and 2π.

- Here the spectral estimation task may be to estimate the frequency of the sine-wave from observations of the process
- For small N the sine-wave component can be hidden in the noise of the periodogram
- As N increases both the frequency resolution and signal-to-noise ratio improve.
- In the case of the random phase sine wave alone, the variance of the periodogram is very small, and hence the errors for small N are mostly due to the bias, which reduces asymptotically to zero.
- See figures below - periodograms for various values of N


N= 16
Periodogram
Normalised frequency, ωT/(2π)


N=64
Periodogram
Normalised frequency, ωT/(2π)


N=128
Periodogram
Normalised frequency, ωT/(2π)



120
100
80
60
40
20
0
Periodogram
N=1024
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
Normalised frequency, ωT/(2π)