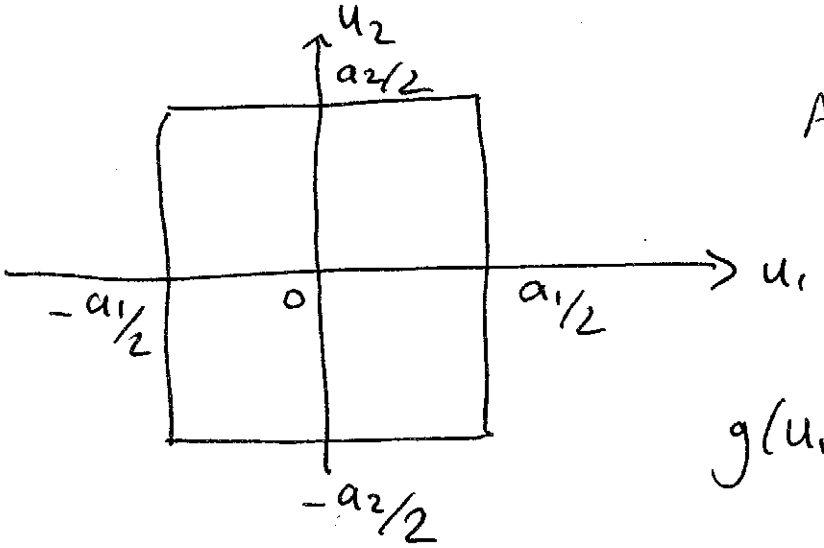
Image Processing 4F8 – Example Sheet

①



Assume black = 0
white = A (const)

$$g(u_1,u_2) = \begin{cases} A & |u_1| < a_1/2 \text{ and } |u_2| < a_2/2 \\ 0 & \text{otherwise} \end{cases}$$

Take FT of this image:

$$G(\omega_1,\omega_2) = \int_{-\infty}^{\infty}\!\!\int g(u_1,u_2)\, e^{-j(\omega_1 u_1 + \omega_2 u_2)}\, du_1\, du_2$$

$$= A \int_{u_2=-a_2/2}^{a_2/2}\int_{u_1=-a_1/2}^{a_1/2} e^{-j(\omega_1 u_1 + \omega_2 u_2)}\, du_1\, du_2 = A\int_{-a_1/2}^{a_2/2} e^{-j\omega_1 u_1}\, du_1 \int_{-a_2/2}^{a_2/2} e^{-j\omega_2 u_2}\, du$$

$$= A\left[\frac{e^{-j\omega_1 u_1}}{-j\omega_1}\right]_{-a_1/2}^{a_1/2}\left[\frac{e^{-j\omega_2 u_2}}{-j\omega_2}\right]_{-a_2/2}^{a_2/2}$$

$$= A a_1 a_2 \left[\frac{e^{ja_1\omega_1/2} - e^{-ja_1\omega_1/2}}{2j\, \omega_1 a_1/2}\right]\left[\frac{e^{ja_2\omega_2/2} - e^{-ja_2\omega_2/2}}{2j\, \omega_2 a_2/2}\right]$$

$$\Rightarrow \boxed{G(\omega_1, \omega_2) = A a_1 a_2 \operatorname{sinc}\frac{a_1\omega_1}{2}\operatorname{sinc}\frac{a_2\omega_2}{2}}$$

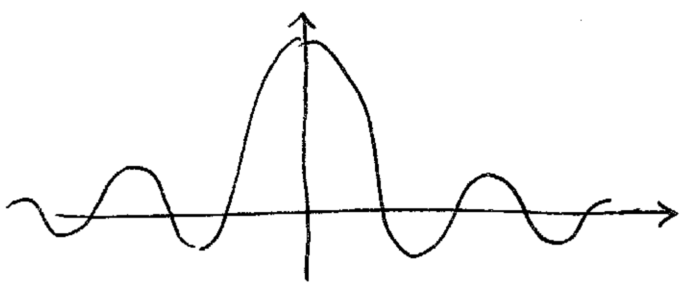
(1) cont.. Now suppose our white rectangle is shifted so that it is centred at (μ_1, μ_2). We know that a shift in spatial domain $\Rightarrow$ complex phase factor in the frequency domain according to

$$g(u_1 - \mu_1, u_2 - \mu_2) \Leftrightarrow e^{-j(\mu_1\omega_1 + \mu_2\omega_2)} G(\omega_1, \omega_2)$$

$\therefore$ New FT is

$$\boxed{G'(\omega_1, \omega_2) = A a_1 a_2 \operatorname{sinc}\frac{a_1\omega_1}{2} \cdot \operatorname{sinc}\frac{a_2\omega_2}{2} e^{-j(\mu_1\omega_1 + \mu_2\omega_2)}}$$

Sinc function looks like (in ω_1 direction)



~~1st~~ zeros occur at $\sin\frac{a_1\omega_1}{2} = 0$

$\therefore$ 1st zero occurs at

$\frac{a_1\omega_1}{2} = \pi$ or $\omega_1 = \pm\frac{2\pi}{a_1}$

Similarly in ω_2 direction, first zero occurs at

$\omega_2 = \pm\frac{2\pi}{a_2}$ $\Rightarrow$ 'bandwidths' are $\underbrace{\frac{2\pi}{a_1}}_{\omega_1}, \underbrace{\frac{2\pi}{a_2}}_{\omega_2}$

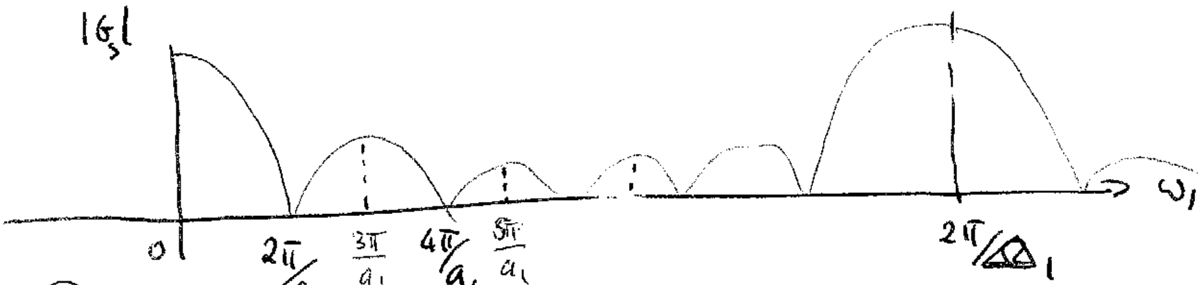
[Could use other definitions of bandwidth, eg 3dB down etc...]

2) Above image is sampled on a rectangular grid : let sampling be Δ_1, Δ_2 in u_1 and u_2 directions. Know that the FT of a sampled image, g_s, is simply proportional to the periodic repetition of the unsampled FT – more specifically:

$$G_s(\omega_1, \omega_2) = \frac{1}{\Delta_1 \Delta_2} \sum_{p_1=-\infty}^{\infty} \sum_{p_2=-\infty}^{\infty} G(\omega_1 - p_1\Omega_1, \omega_2 - p_2\Omega_2)$$

where $\Omega_i = \dfrac{2\pi}{\Delta_i}$ – see P. ~~4~~ Handout 2.

$\therefore$ in ω_1 direction we have



zeros occur at ~~sin~~ $\dfrac{a\omega_1}{2} = n\pi \quad \therefore \omega_1 = \dfrac{2n\pi}{a_1}$ $\quad \left(\sin \dfrac{\omega_1 a}{2} = 0\right)$

Since spectrum is not strictly bandlimited we will always have aliasing. Investigate the amplitude of the other sidebands . . .

$a_1 \operatorname{sinc} \dfrac{a_1 w_1}{2} \Rightarrow$

1st $\quad a_1 \operatorname{sinc} \dfrac{3\pi}{a_1} \cdot \dfrac{a_1}{2} = -a_1 0.2122$

2nd $\quad a_1 \operatorname{sinc} \dfrac{5\pi}{a_1} \cdot \dfrac{a_1}{2} = a_1 0.1273$

3rd $\quad a_1 \operatorname{sinc} \dfrac{7\pi}{a_1} \cdot \dfrac{a_1}{2} = a_1 0.0909$

4th $\quad a_1 \operatorname{sinc} \dfrac{9\pi}{a_1} \cdot \dfrac{a_1}{2} = a_1 0.0707$

2 cont..

⋮

$$10^{th} \quad a_1 \operatorname{sinc} \frac{21\pi}{a_1} \cdot \frac{a_1}{a} = a_1\, 0.0303$$

Recall that we want to find the sideband which is 30dB down on the mainlobe:

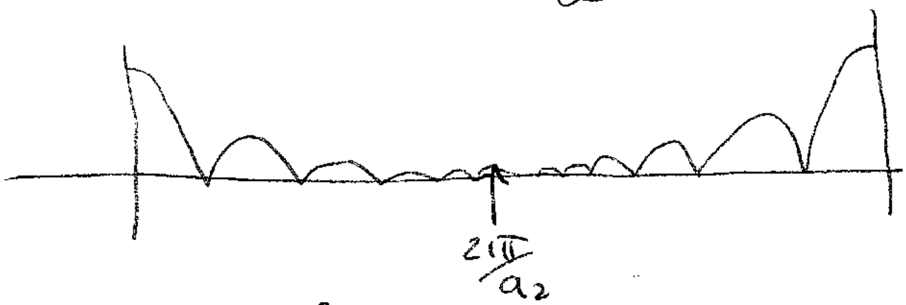
$$20 \log_{10} \frac{x_1}{x_2} = -30 \quad \therefore \quad \log_{10} \frac{x_1}{x_2} = \frac{-3}{2}$$

(amplitude, not power)

$$\therefore \quad x_1 = x_2\, 10^{-3/2} = \frac{x_2}{\sqrt{1000}} = 0.0316\, x_2$$

$\therefore$ the sideband which is approx 30dB down from mainlobe is at (10th sideband.)

$$\frac{\omega_1 a_1}{2} = \frac{21\pi}{2} \qquad \therefore \omega_1 = \frac{21\pi}{a_1}.$$



Clearly we require the sampling to be at least twice this value for the 'aliased components' to be 30dB down.

$$\Rightarrow \quad \frac{\cancel{2}\pi}{\Delta_1} > \cancel{2} \cdot \frac{21\pi}{a_1} \qquad \therefore \Delta_1 < \frac{a_1}{21}$$

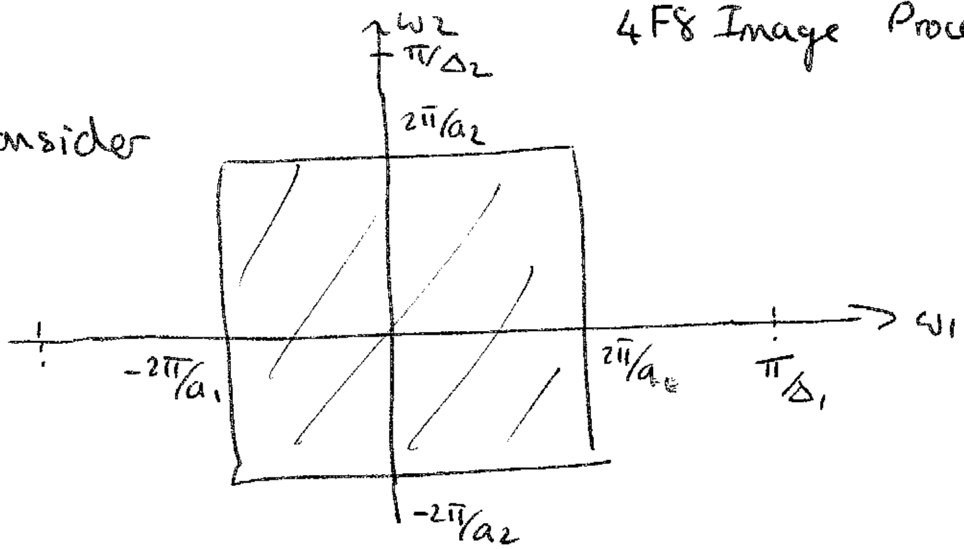
A similar argument for ω_2 direction yields $\Delta_2 < \dfrac{a_2}{21}$

$\Rightarrow$ sample spacings required are

$$\boxed{\Delta_1 < \frac{a_1}{21} \quad \text{and} \quad \Delta_2 < \frac{a_2}{21}}$$

(Some variation OK)

(3c) Consider



$$H = \begin{cases} 1 & |\omega_1| < 2\pi/a_1 \text{ and } |\omega_2| < \frac{2\pi}{a_2} \\ 0 & \text{otherwise.} \end{cases}$$

If the sampling in u_1 and u_2 is Δ_1, Δ_2 then we have seen (P.2 H3) that

$$h(n_1\Delta_1, n_2\Delta_2) = \frac{1}{\Omega_1\Omega_2} \int_{-\Omega_2/2}^{\Omega_2/2}\int_{-\Omega_1/2}^{\Omega_1/2} H_s(\omega_1, \omega_2)\, e^{j(\omega_1 n_1\Delta_1 + \omega_2 n_2\Delta_2)}\, d\omega_1 d\omega_2$$

$\Omega_i = 2\pi/\Delta_i$

$$= \frac{\Delta_1\Delta_2}{(2\pi)^2} \int_{-2\pi/a_2}^{2\pi/a_2}\int_{-2\pi/a_1}^{2\pi/a_1} e^{j(\omega_1 n_1\Delta_1 + \omega_2 n_2\Delta_2)}\, d\omega_1 d\omega_2$$

$$= \frac{\Delta_1\Delta_2}{(2\pi)^2}\, 2\left[\frac{e^{j\omega_1 n_1\Delta_1}}{2j\, n_1\Delta_1 \frac{2\pi}{a_1}}\right]_{-2\pi/a_1}^{2\pi/a_1} \frac{2\pi}{a_1}\; 2\left[\frac{e^{j\omega_2 n_2\Delta_2}}{2j\, n_2\Delta_2 \frac{2\pi}{a_2}}\right]_{-2\pi/a_2}^{2\pi/a_2} \frac{2\pi}{a_2}$$

$$= \boxed{\frac{4\Delta_1\Delta_2}{a_1 a_2}\, \text{sinc}\, \frac{2\pi n_1\Delta_1}{a_1}\, \text{sinc}\, \frac{2\pi n_2\Delta_2}{a_2}}$$

c.f expression on P.15 H3 for 'normalised frequencies'

$$h(n_1, n_2) = \frac{1}{\pi^2}\, \frac{2\pi}{a_1} \cdot \frac{2\pi}{a_2}\, \text{sinc}\, \frac{2\pi n_1}{a_1}\, \text{sinc}\, \frac{2\pi n_2}{a_2}$$

— see that if replace $a_i \to a_i/\Delta_i$ we recover answer.

3 cont..

Note this is a separable filter but has infinite support. To make it into an FIR filter we window – with a separable window

$$h_w(n_1, n_2) = h(n_1, n_2)\, w(n_1, n_2)$$

$$= [h_1(n_1) w_1(n_1)][h_2(n_2) w_2(n_2)]$$

separable.

Where (say) $w_i(n_i) = \begin{cases} 1 & |n_i| < M_i \\ 0 & \text{otherwise.} \end{cases}$

Thus, the filtering operation, which is a 2D convolution, is given by

$$y(n_1, n_2) = \sum_{m_2=-M_2}^{M_2} \sum_{m_1=-M_1}^{M_1} \underbrace{h_1'(m_1)}_{h_i' = h_i(n_i) w_i(n_i)} h_2'(m_2)\, x(n_1 - m_1, n_2 - m_2)$$

Let image be of dimensions N_1, N_2

$$= \sum_{m_2=-M_2}^{M_2} h_2'(m_2) \sum_{m_1=-M_1}^{M_1} h_1'(m_1)\, x(n_1 - m_1, n_2 - m_2)$$

For this <u>separable filter</u> we first do the row filtering requiring (for a given m_2) $N_1 N_2 (2M_1 + 1)$ operations

Then, for each column we do a further $N_1 N_2 (2M_2 + 1)$ operations

$\Rightarrow$ total is $\boxed{N_1 N_2 [(2M_1 + 1) + (2M_2 + 1)]}$

Q3 continued:

For a non-separable filter we have

$$y(n_1,n_2) = \sum_{-M_2}^{M_2} \sum_{-M_1}^{M_1} h(m_1,m_2)x(n_1-m_1, n_2-m_2)$$

so that for a given (n_1,n_2) we require $(2M_1+1)(2M_2+1)$ operations

$$\Rightarrow \boxed{N_1 N_2 (2M_1+1)(2M_2+1)}$$

operations are required for the whole imge $\Rightarrow$ greater computational load, Q4

Have seen (handout 2) that provided an image, $x(u_1,u_2)$, has been sampled at or above the Nyquist frequency, we can completely recover x from its samples via

$$x(u_1,u_2) = \sum_{n_1}\sum_{n_2} x(n_1\Delta_1, n_2\Delta_2)\,\mathrm{sinc}\left(\frac{u_1-n_1\Delta_1}{\Delta_1}\right)\pi$$

$$\times\ \mathrm{sinc}\left(\frac{u_2-n_2\Delta_2}{\Delta_2}\right)\pi$$

∴ It's clear how we should form our interpolated image:

$$x\left(\frac{p_1\Delta_1}{2}, \frac{p_2\Delta_2}{2}\right) = \sum_{n_1=-\infty}^{+\infty}\sum_{n_2=-\infty}^{+\infty} x(n_1\Delta_1, n_2\Delta_2)\,\mathrm{sinc}\left(\frac{p_1\Delta_1}{2}-n_1\Delta_1\right)\frac{\pi}{\Delta_1}$$

$$\times\ \mathrm{sinc}\left(\frac{p_2\Delta_2}{2}-n_2\Delta_2\right)\frac{\pi}{\Delta_2}$$

$$\Rightarrow x\left(\frac{p_1\Delta_1}{2}, \frac{p_2\Delta_2}{2}\right) = \sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty} x(n_1\Delta_1, n_2\Delta_2)\,\mathrm{sinc}\,(p_1-2n_1)\frac{\pi}{2}$$

$$\times\ \mathrm{sinc}\,(p_2-2n_2)\frac{\pi}{2}$$

Note in practice a sinc interpolater may be too computationally expensive – may wish to use something simpler (linear, quadratic etc)

Question 5:

1. To find the spectrum of the 2d cosine window formed from the product of two 1d windows, first find the FT of w_1

$$W_1(\omega_1) = \int_{-U_1}^{U_1} \cos\left(\frac{\pi u_1}{U_1}\right) \mathrm{e}^{-j\omega_1 u_1} du_1$$

$$= \frac{1}{2}\int_{-U_1}^{U_1} \mathrm{e}^{ju_1(\pi/U_1-\omega_1)} + \mathrm{e}^{-ju_1(\pi/U_1+\omega_1)} du_1$$

$$= \frac{1}{2}\left[\frac{\mathrm{e}^{ju_1(\pi/U_1-\omega_1)}}{j(\pi/U_1-\omega_1)} - \frac{\mathrm{e}^{-ju_1(\pi/U_1+\omega_1)}}{j(\pi/U_1+\omega_1)}\right]_{-U_1}^{U_1}$$

$$= U_1\{\ \mathrm{sinc}(\pi-\omega_1 U_1) + \ \mathrm{sinc}(\pi+\omega_1 U_1)\} \tag{1}$$

As before, $W(\omega_2)$ will take precisely the same form so that the required spectrum will be the product of W_1 and W_2.

$$W(\omega_1,\omega_2) = U_1 U_2\{\ \mathrm{sinc}(\pi-\omega_1 U_1) + \ \mathrm{sinc}(\pi+\omega_1 U_1)\}\{\ \mathrm{sinc}(\pi-\omega_2 U_2) + \ \mathrm{sinc}(\pi+\omega_2 U_2)\}$$

Spectrum along ω_1 axis looks like:

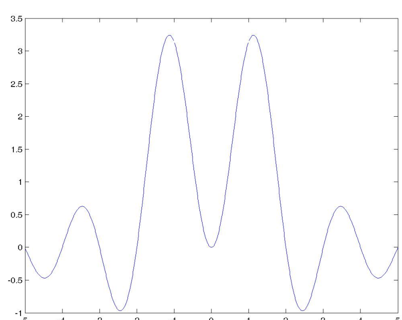


Figure 1: The spectrum of the 1d cosine window: drawn with $U_1 = \pi$

As we can see from the above plot, the spectrum of the cosine window has a wide main lobe with a significant depression at $\omega = 0$ – even though the sidelobes are fairly low, the mainlobe characteristics are not desirable.

2. Now find the spectrum of the 2d window formed from the product of two 1d rectangular windows, where we now have

$$w_i(u_i) = \begin{cases} 1 & \text{if } |u_i| < U_i \\ 0 & \text{otherwise} \end{cases}$$

First find the FT of w_1

$$W_1(\omega_1) = \int_{-U_1}^{U_1} \mathrm{e}^{-j\omega_1 u_1} du_1$$

$$= \left[\frac{\mathrm{e}^{-j\omega_1 u_1}}{-j\omega_1} \right]_{-U_1}^{U_1}$$

$$= 2U_1 \ \mathrm{sinc}\omega_1 U_1$$

.

$W(\omega_2)$ will take precisely the same form so that the required spectrum will be the product of W_1 and W_2.

$$W(\omega_1, \omega_2) = 4U_1U_2 \ \mathrm{sinc}\omega_1 U_1 \ \mathrm{sinc}\omega_2 U_2$$

Spectrum looks like:

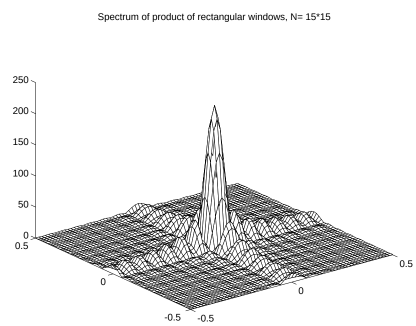


Figure 2:

where, U_1 is (for illustrative purposes) taken as 2.5π in the above sketch and units on ω_1 and ω_2 axes are in units of 2π.

3. We can deduce the spectrum of this superposition of windows from the above results:

$$W_1(\omega_1) = U_1 \left(2\alpha \ \mathrm{sinc}\omega_1 U_1 + \beta\{ \ \mathrm{sinc}(\pi - \omega_1 U_1) + \ \mathrm{sinc}(\pi + \omega_1 U_1)\}\right)$$

And similarly for $W_2(\omega_2)$.

If we plot W_1 (doing it in 1d will do) while varying α (using $\alpha + \beta = 1$), we can see what happens to the spectrum – some examples are given in figure 3 and figure 4. Figure 4 is the optimal value of α, ie the value which causes the first and largest sidelobes to be suppressed.

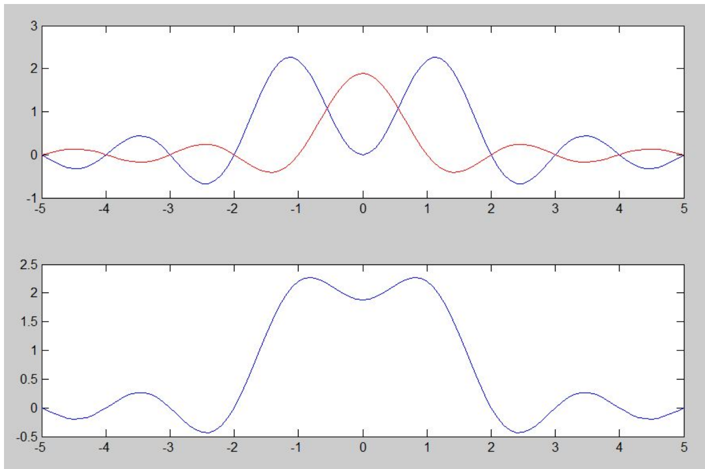


Figure 3: Upper graph shows the spectra of β times the cosine window and α times the rectangular window for $\alpha = 0.3$. The lower graph shows the resulting superposition.

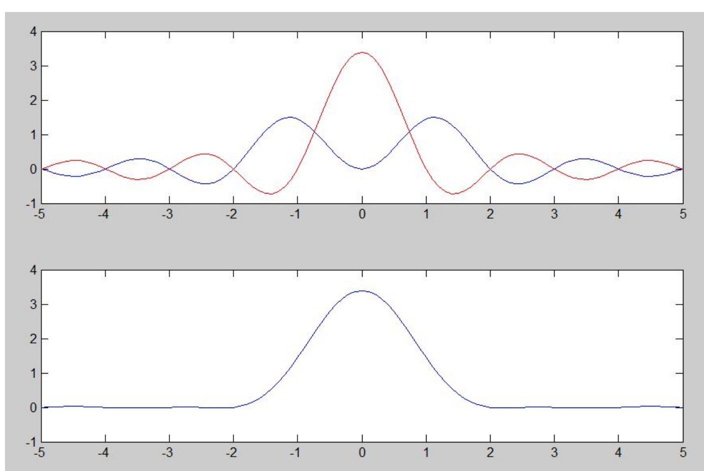


Figure 4: Upper graph shows the spectra of β times the cosine window and α times the rectangular window for $\alpha = 0.54$. The lower graph shows the resulting superposition.

Question 6:

1. First read in the colour image *moireB.jpg* take just one of the channels and then downsample by a factor of 2 in both dimensions: do this using something like the following code:

```
% input the name of the image to test
s1 = input('Filename:', 's');
s1=['/jl/4F8imageprocessing/2009-10/examplespaper/',s1]; % put where it is
A =imread(s1);
figure(1)
imshow(X);
% take just the first channel of this colour image
A1 = double(X(:,:,1));
ndown=2;
C1 = downsample(A1,ndown);
Cds = downsample(C1',ndown);
figure(2); grayimage(Cds);
```

The original and the downsampled image are show in figure 1:

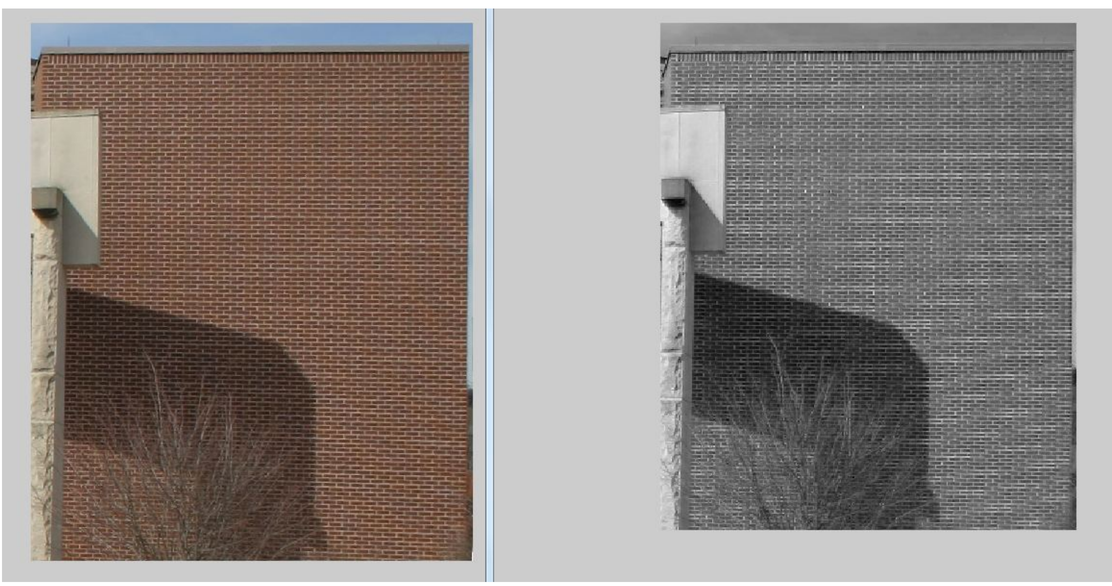


Figure 1: Left hand figure shows the orginal 756×622 colour image, *moireB.jpg*. The right hand figure shows the first channel of this image downsampled by 2 (size is then 378×311)

Aliasing artefacts are visible in the right-hand image of figure 1.

2. Next we take FFTs of both the original (first channel) and downsampled images. These are shown in figure 2.

   Figure 3 shows a comparison of regions of roughly the same frequency range and highlights some of the visible aliasing.

3. Let us suppose that our continuous image has lengths a_1 (horizontal) and a_2 (vertical). For the original image the spacings are therefore

$$\Delta_1 = a_1/622 \quad \Delta_2 = a_2/756$$

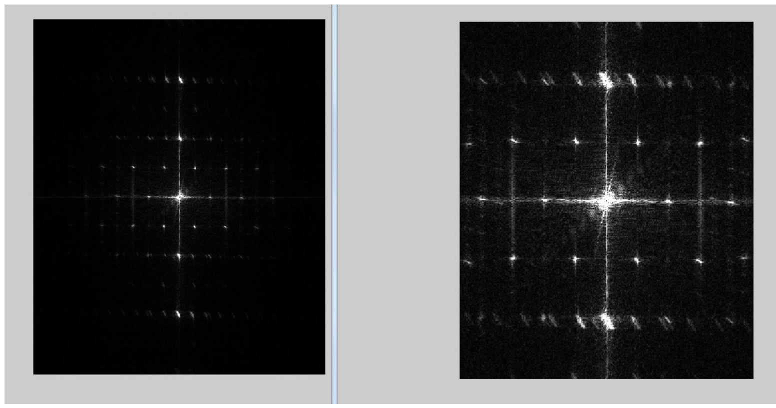


Figure 2: Left hand figure shows the FFT of the original image (channel 1). The right hand figure shows the FFT of the downsampled image

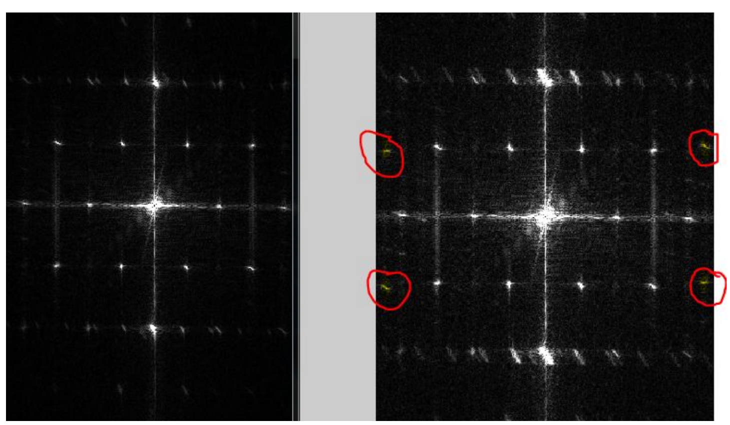


Figure 3: A comparison of similar frequency ranges for the original and downsampled images. The right-hand image has some of the aliased frequencies outlined in red.

We assume that this is an unaliased image. ie that $\Omega_1 > 2\Omega_{C1}$ and $\Omega_2 > 2\Omega_{C2}$, where $\Omega_1 = 2\pi/\Delta_1$ and $\Omega_2 = 2\pi/\Delta_2$, and Ω_{C1} and Ω_{C2} are the highest frequencies in the image.

From our FFT of A, we can estimate the largest frequencies in the image, see figure 4:

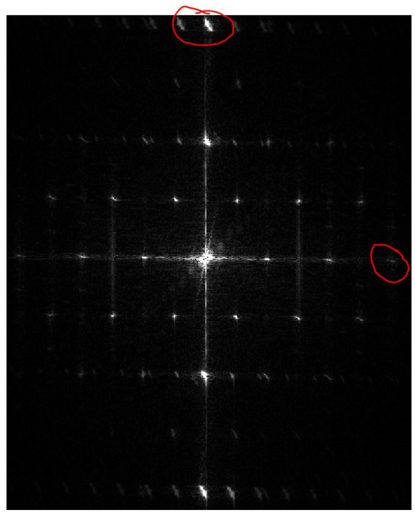


Figure 4: Original spectrum with approx highest frequencies outlined in red

We see that the highest ω_1 frequency is at approx $(512 - 312) = 200$ and the highest ω_2 frequency is at approx $(378 - 128) = 250$. Thus, we approximate the largest directional frequencies as

$$\Omega_{C1} \approx 200 \times (2\pi/a_1) \qquad \Omega_{C2} \approx 250 \times (2\pi/a_2)$$

So, suppose we sample at Δ_{1a} and Δ_{2a} – for no aliasing, we then require

$$\frac{2\pi}{\Delta 1a} > 2\Omega_{C1} = \frac{800\pi}{a_1}$$

and

$$\frac{2\pi}{\Delta 2a} > 2\Omega_{C2} = \frac{1000\pi}{a_2}$$

But $\Delta_{1a} = a_1/n_1$ and $\Delta_{2a} = a_2/n_2$, so we have that

$$\frac{a_1}{n_1} < \frac{a_1}{400} \qquad \frac{a_1}{n_1} < \frac{a_1}{500}$$

Thus, we need $n_1 > 400$ and $n_2 > 500$ – so a 500×400 image would be the minimum needed for no aliasing.

Question 7:

1. Can create the 512×512 images of black and white stripes (0 and 255) and either write code to rotate the central 256×256 image, or use the *imrotate* command in Matlab. Figure 1 shows the central image B and the version rotated by 7 degrees.

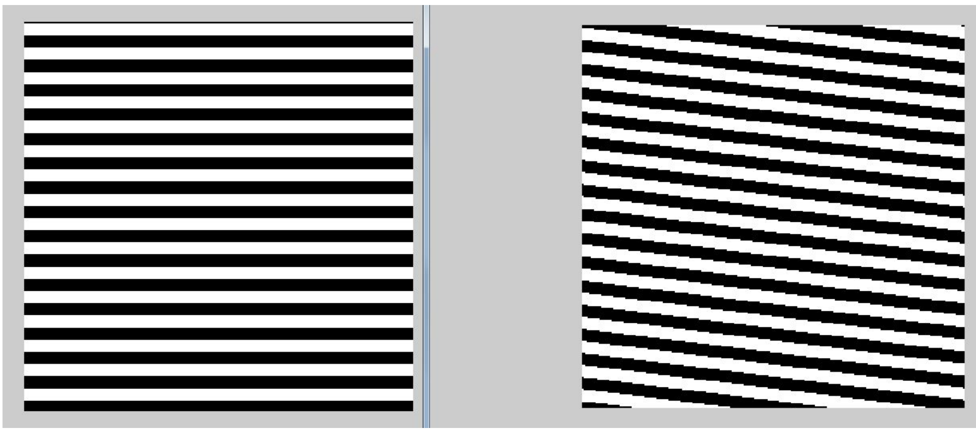


Figure 1: Left hand figure shows image B, with stripes 8 pixels wide. The right hand figure shows this image rotated clockwise by 7 degrees

2. Adding these two images together (rescale so that resultant goes from 0 to 255) gives image C shown in figure 2:

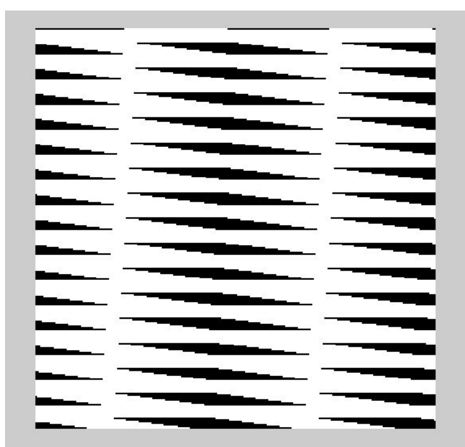


Figure 2: Addition of the two images in figure 1

Note the interference patterns when the two images are added, giving fringes of specific frequencies.

3. Figure 3 shows the FFTs of each of the images B, rotated B and C:

We can see that along the central 'vertical', the addition produces frequencies which are very close to each other. The closeness of these frequency components, as we have seen, will produce 'beating', ie sum and difference effects, which will manifest themselves as interference patterns.

From image C it is clear that the vertical spacing of the pattern is 16 (there are 16 repetitions in the 256 length) and that the horizontal spacing is 128 (there are two repetitions in the 256 width). Thus, we would expect that these arise from frequency differences of 16 in vertical frequency (from $1/16 = n/256$) and 2 in horizontal frequency (from $1/128 = n/256$). Figure 4 indicates the two

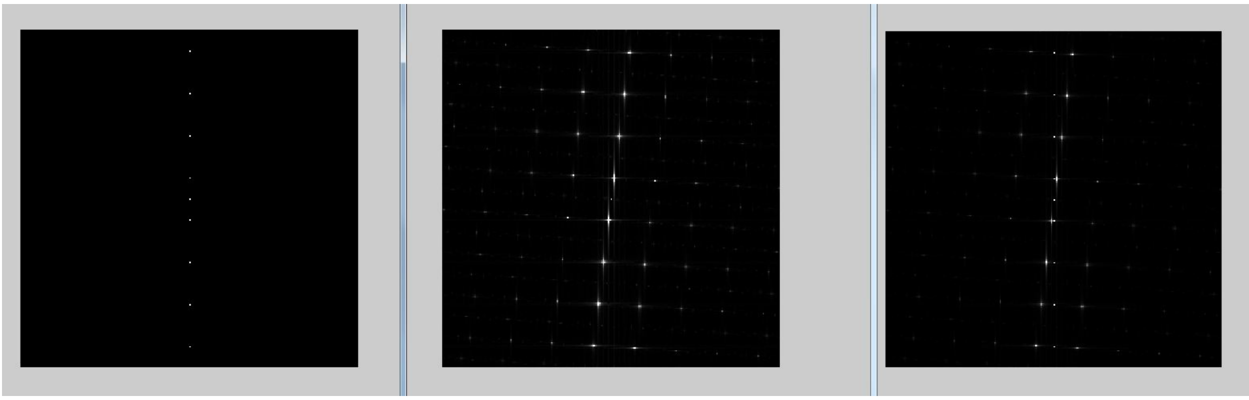


Figure 3: Frequency spectra image B [left], rotated B [centre], and image C [right]

closely spaced frequencies near the centre of the frequency plane which will give rise to this (at points (129,129) and (131,113) in the 256×256 frequency plane).

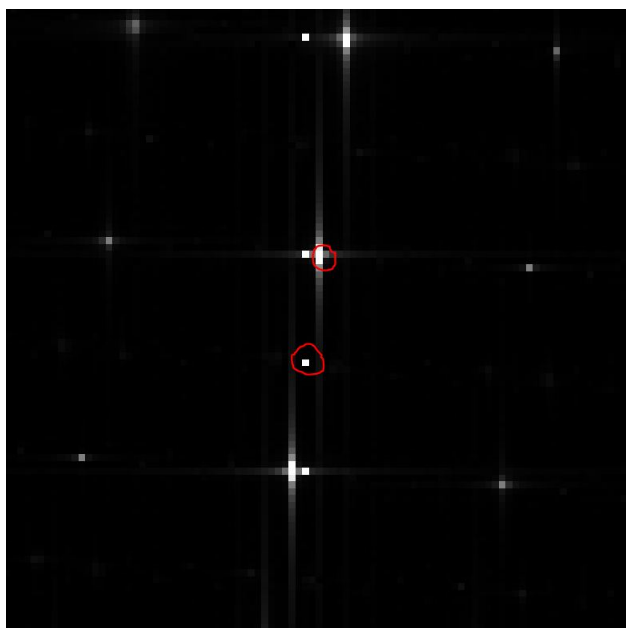


Figure 4: The two main frequencies which give rise to the interference are indicated in red

6)

$$\cancel{R_{xx}(q_1, q_2)} \quad R_{xx}(n_1, n_2) = \sigma_g^2 e^{-(\alpha_1|n_1| + \alpha_2|n_2|)}$$

Know that the power spectrum is the FT of the autocorrelation function, ie

$$P_{xx}(\omega_1, \omega_2) = \sum_{n_2=-\infty}^{+\infty}\sum_{n_1=-\infty}^{+\infty} R_{xx}(n_1, n_2)\, e^{-j(\omega_1 n_1 \Delta_1 + \omega_2 n_2 \Delta_2)}$$

(Handout 3 p.2)

Here we are assuming that $R_{xx}(n_1, n_2) \equiv R_{xx}(n_1\Delta_1, n_2\Delta_2)$

$$\therefore P_{xx} = \sum_{n_2}\sum_{n_1} \sigma_g^2\, e^{-\alpha_1|n_1| - j\omega_1 n_1 \Delta_1} \cdot e^{-\alpha_2|n_2| - j\omega_2 n_2 \Delta_2}$$

$$= \sigma_g^2 \sum_{n_1} e^{-[\alpha_1|n_1| + j\omega_1 n_1 \Delta_1]} \sum_{n_2} e^{-[\alpha_2|n_2| + j\omega_2 n_2 \Delta_2]}$$

Consider one of these terms $\displaystyle\sum_{n_1=-\infty}^{\infty} e^{-[\alpha_1|n_1| + j\omega_1 n_1 \Delta_1]}$

$$= \sum_{n_1=-\infty}^{-1} e^{\alpha_1 n_1 - j\omega_1 n_1 \Delta_1} + \underset{(n_1=0)}{1} + \sum_{n_1=1}^{\infty} e^{-\alpha_1 n_1 - j\omega_1 n_1 \Delta_1}$$

$$\cancel{= 1 + 2\sum_{n_1=1}^{\infty} e^{-\alpha_1 n_1} \left[\frac{e^{j\omega_1 n_1 \Delta_1} + e^{-j\omega_1 n_1 \Delta_1}}{2}\right]} =$$

6) cont..

$$\cancel{= 1 + 2\sum_{n_1=1}^{\infty} e^{-\alpha_1 n_1} \cos \omega_1 n_1 \Delta_1}$$

$$\therefore \; P_{xx}\Big|_{\text{term } 1} = 1 + \sum_{n_1=1}^{\infty} e^{-\alpha_1 n_1 + j\omega_1 n_1 \Delta_1} + \sum_{n_1=1}^{\infty} e^{-\alpha_1 n - j\omega_1 n_1 \Delta_1}$$

$$= 1 + \sum_{n_1=1}^{\infty} e^{n_1(j\omega_1\Delta_1 - \alpha_1)} + \sum_{n_1=1}^{\infty} e^{-n_1(j\omega_1\Delta_1 + \alpha_1)}$$

Sum these since they are GP's $\quad S = \sum_{r=1}^{\infty} a^r \quad |a| < 1$

$$S = a + a^2 + a^3 + \cdots$$
$$aS = a^2 + a^3 + \cdots$$
$$\therefore S(1-a) = a \quad \therefore S = \frac{a}{1-a}$$

$$\therefore \; P_{xx}(\text{term } 1) = 1 + \frac{e^{j\omega_1\Delta_1 - \alpha_1}}{1 - e^{j\omega_1\Delta_1 - \alpha_1}} + \frac{e^{-(j\omega\Delta_1 + \alpha_1)}}{1 - e^{-(j\omega_1\Delta_1 + \alpha_1)}}$$

$$= 1 + \frac{1}{e^{\alpha_1 - j\omega_1\Delta_1} - 1} + \frac{1}{e^{\alpha_1 + j\omega_1\Delta_1} - 1} = \frac{1 + e^{\alpha_1 + j\omega_1\Delta_1} - 1 + e^{\alpha_1 - j\omega_1\Delta_1} - 1}{(\quad)(\quad)}$$

$$= 1 + \frac{e^{\alpha_1}\left(e^{j\omega_1\Delta_1} + e^{-j\omega_1\Delta_1}\right) - 2}{e^{2\alpha} - e^{\alpha}\left(e^{j\omega_1\Delta_1} + e^{-j\omega_1\Delta_1}\right) + 1} =$$

6 cont

$$= 1 + \frac{[2e^{\alpha_1}\cos\omega_1\Delta_1 - 2]e^{-\alpha}}{[\underbrace{e^{2\alpha}+1} - 2e^{\alpha}\cos\omega_1\Delta_1]e^{-\alpha}}$$

$$= 1 + \frac{\cancel{2}\cos\omega_1\Delta_1 - \cancel{2}e^{-\alpha}}{\cancel{2}\cosh\alpha_1 - \cancel{2}\cos\omega_1\Delta_1}$$

$$= \frac{\cosh\alpha_1 - \cancel{\cos\omega_1\Delta_1} + \cancel{\cos\omega_1\Delta_1} - e^{-\alpha}}{\cosh\alpha_1 - \cos\omega_1\Delta_1}$$

but $\cosh\alpha_1 = \dfrac{e^{\alpha_1}+e^{-\alpha_1}}{2}$ $\therefore \cosh\alpha_1 - e^{-\alpha_1} = \sinh\alpha_1$

$$\therefore = \frac{\sinh\alpha_1}{\cosh\alpha_1 - \cos\omega_1\Delta_1}$$

$$\Rightarrow \boxed{P_{xx}(\omega_1\,\omega_2) = \sigma_g^2\left[\frac{\sinh\alpha_1}{\cosh\alpha_1 - \cos\omega_1\Delta_1}\right]\left[\frac{\sinh\alpha_2}{\cosh\alpha_2 - \cos\omega_2\Delta_2}\right]}$$

Wiener filter is derived in lectures (either way)

$$G(\omega_1\,\omega_2) = \frac{H^*(\underline{\omega})\,P_{xx}(\underline{\omega})}{|H(\underline{\omega})|^2 P_{xx}(\underline{\omega}) + P_{dd}(\underline{\omega})} \qquad H = I$$

$$\therefore G(\omega_1,\omega_2) = \frac{P_{xx}(\underline{\omega})}{P_{xx}(\underline{\omega}) + P_{dd}(\underline{\omega})}$$

(6) cont...

But, we are given that we have zero mean additive white noise : white $\Rightarrow$ P_{dd} is uniform over all frequencies and $P_{dd} = \sigma_g^2$

$$\Rightarrow \quad G(\underline{\omega}) = \frac{P_{xx}(\underline{\omega})}{P_{xx}(\underline{\omega}) + \sigma_g^2}$$

[Stationary white noise process has

$$R_{xx}(n_1, n_2) = \sigma_x^2 \delta(n_1, n_2)$$

$$P_{xx}(\omega_1 \omega_2) = FT(R_{xx}) = \sigma_x^2 \quad \forall \underline{\omega} \; .]$$