Communications

IB Paper 6

Handout 4: Digital Modulation and Bits Through Channels

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Lent Term

Outline

1. Digital Modulation
   - Amplitude Modulation
   - Phase Modulation
   - Frequency Modulation
2. Practical Examples
   - The GSM Modulation
   - Multicarrier Modulation
3. Bits Through Channels
   - Error Probability and Rate
   - Shannon's Theorem

Introduction and Motivation

Recap from Handout 3

We represent digital signals as a pulse train

$$x(t) = \sum_k a_k p(t - kT)$$

- a_k is the k-th bit (or more in general, symbol) in the message sequence, $a_k = \pm\frac{A}{2}$
- T is the symbol period
- $p(t)$ is the pulse such that

$$p(t) = \begin{cases} 1 & t = 0 \\ 0 & t = \pm T, \pm 2T, \ldots \end{cases}$$

- Spectrum of unmodulated signal $|X(f)|^2 = \frac{1}{T}|P(f)|^2$

Digital Modulation

Introduction

- We will use digital modulation techniques when the information signal is digital
- Again the information signal modulates the amplitude, phase or frequency of a carrier wave

$$s(t) = a\cos(2\pi f_c t + \phi)$$

- A few interesting differences wrt analog modulation
- For example, phase digital modulation widely employed
- The BSC (see end of this Handout) summarises digital modulation and demodulation effects through ϵ, the probability of the channel giving an output in error

Digital Modulation

ASK Modulation

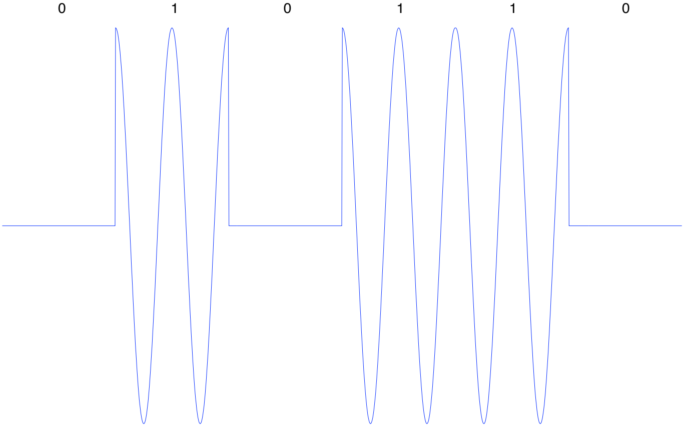
Amplitude Shift Keying (ASK)

ASK is the digital counterpart of AM. Considering binary ASK only we have

$$s_{\text{ASK}}(t) = \begin{cases} a\cos(2\pi f_c t) & \text{for information bit 1} \\ 0 & \text{for information bit 0} \end{cases}$$

Digital Modulation

ASK Modulation



Phase Shift Keying (PSK)

PSK is the digital counterpart of PM. Considering binary PSK (BPSK) only we have

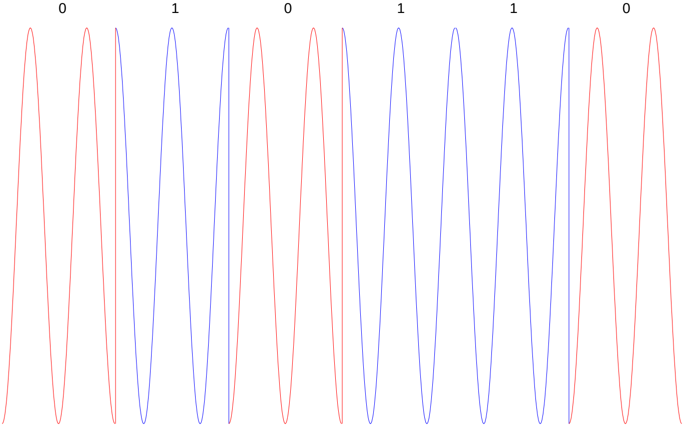
$$s_{\mathrm{PSK}}(t) = \begin{cases} a\cos(2\pi f_c t) & \text{for information bit 1} \\ a\cos(2\pi f_c t + \pi) & \text{for information bit 0} \end{cases}$$

Note that a phase shift of π corresponds to a change of sign

$$s_{\mathrm{PSK}}(t) = \begin{cases} a\cos(2\pi f_c t) & \text{for information bit 1} \\ -a\cos(2\pi f_c t) & \text{for information bit 0} \end{cases}$$

Digital Modulation

PSK Modulation



Digital Modulation

PSK Modulation

Spectrum Calculation

As we have just seen, we can write the BPSK signal as

$$s_{\text{BPSK}}(t) = \underbrace{\sum_k a_k p(t-kT)}_{\text{information signal } x(t)} \underbrace{a\cos(2\pi f_c t)}_{\text{carrier}}$$

From Handout 3, the spectrum of a digital signal is given by

$$|X(f)|^2 = \frac{1}{T}|P(f)|^2$$

where $P(f) = \mathcal{F}[p(t)]$ is the Fourier transform of the pulse $p(t)$.

Spectrum Calculation

Therefore, the BPSK spectrum is

$$S_{\mathrm{BPSK}}(f) = \frac{1}{2}[X(f - f_c) + X(f + f_c)]$$

If $p(t)$ is a rectangular pulse of unit amplitude and duration T, we have that

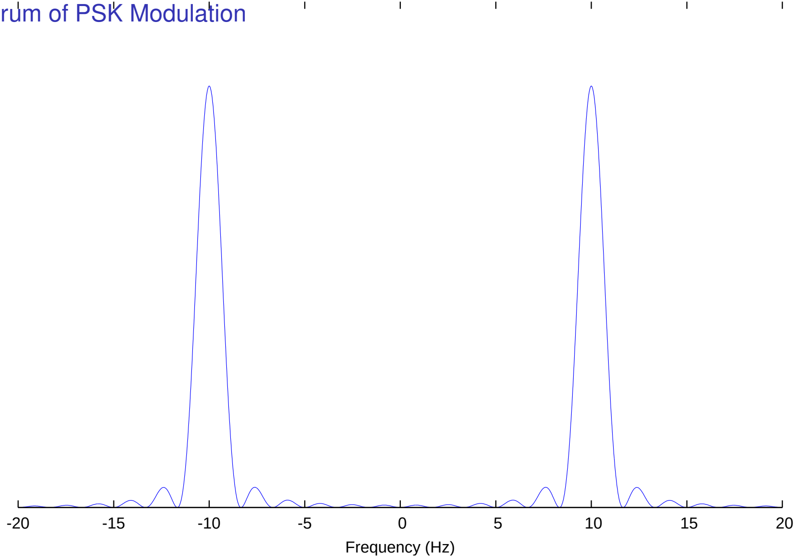
$$P(f) = T\mathrm{sinc}\,(\pi f T) \quad \text{and} \quad |X(f)|^2 = T\mathrm{sinc}^2\,(\pi f T)$$

Assuming no overlap between $X(f - f_c)$ and $X(f + f_c)$ we obtain

$$|S_{\mathrm{BPSK}}(f)|^2 = \frac{1}{4}\left[|X(f - f_c)|^2 + |X(f + f_c)|^2\right]$$

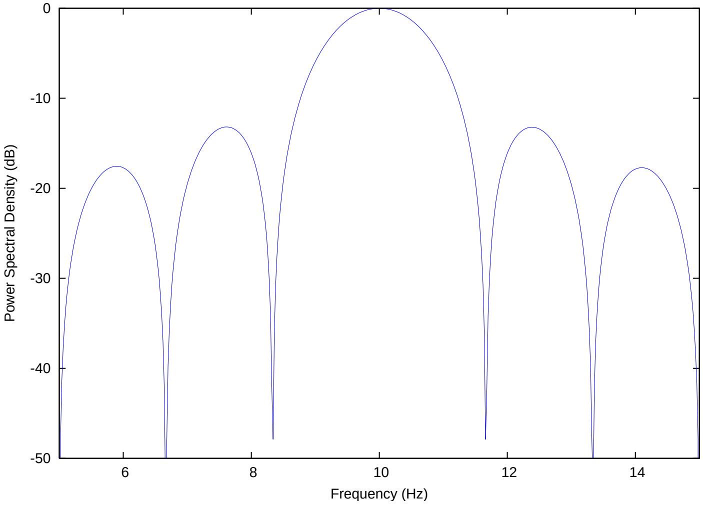
Digital Modulation

Spectrum of PSK Modulation



Digital Modulation

Spectrum of PSK Modulation

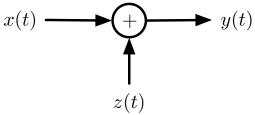


Properties

- As opposed to analogue modulation, strictly speaking, the bandwidth is infinite due to sidelobes
- Usually bandwidth corresponds to the main lobe
- Can also define the bandwidth that contains the 99% of the power

Digital Modulation

Error Probability of BPSK Modulation



Equivalent Digital Model

- Assuming $x(t) = \sum_k a_k p(t - kT)$ (after de-modulation) and sampling when the pulse is at its peak (i.e., $p(t) = 1$) we have
$$Y = X + Z$$
where $X \in \{\pm A\}$ and Z is a Gaussian random variable with zero mean and variance σ^2, i.e., $Z \sim N(0, \sigma^2)$.
- Detection rule
  - if $Y > 0$ decide $\widehat{X} = +A$
  - if $Y < 0$ decide $\widehat{X} = -A$
- We want to calculate the error probability $P_e = p(\widehat{X} \neq X)$

Error Probability

The error probability can be expressed as

$$
\begin{aligned}
P_e &= p(\widehat{X} \neq X) \\
&= p(\widehat{X} = +A | X = -A) p(X = -A) + p(\widehat{X} = -A | X = +A) p(X = +A) \\
&= \frac{1}{2} \left(p(\widehat{X} = +A | X = -A) + p(\widehat{X} = -A | X = +A) \right) \\
&= \frac{1}{2} \left(p(Y > 0 | X = -A) + p(Y < 0 | X = +A) \right) \\
&= p(Y < 0 | X = +A)
\end{aligned}
$$

due to the symmetry of the problem. Conditioned on $X = +A$, Y is a Gaussian random variable, with mean $+A$ and variance σ^2, i.e., $Y \sim N(+A, \sigma^2)$.

Error Probability

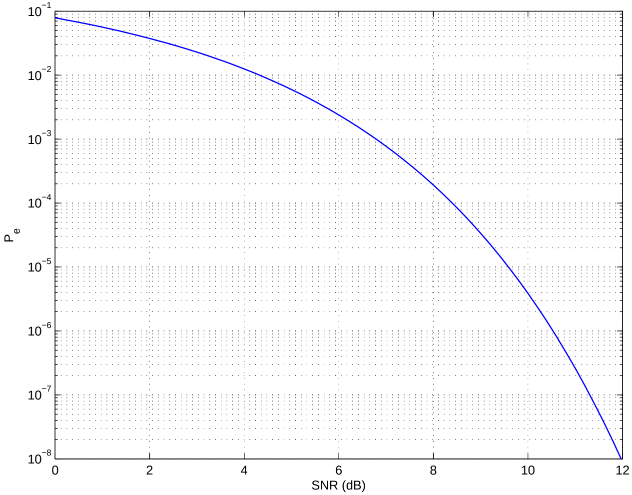
Given that $Y \sim N(+A, \sigma^2)$, the error probability can be expressed as

$$\begin{aligned} P_e = p(Y < 0|X = +A) &= \int_{-\infty}^{0} p_Y(y)dy = \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(y-A)^2} dy \\ &= \int_{-\infty}^{-\frac{A}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du = \Phi\left(-\frac{A}{\sigma}\right) \\ &= Q\left(\frac{A}{\sigma}\right) = Q\left(\sqrt{\frac{A^2}{\sigma^2}}\right) = Q\left(\sqrt{2\mathrm{SNR}}\right) \end{aligned}$$

where $\Phi(x)$ is the Gaussian cumulative (see Probability notes, handout 3), $Q(x) \triangleq 1 - \Phi(x)$ and SNR $\triangleq \frac{A^2}{2\sigma^2}$.

Digital Modulation

Error Probability of BPSK Modulation

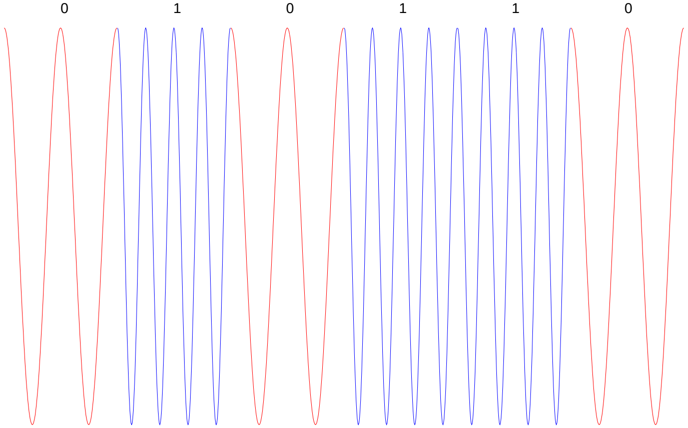


Frequency Shift Keying (FSK)

FSK is the digital counterpart of FM. Considering binary FSK only we have

$$s_{\text{FSK}}(t) = \begin{cases} a\cos(2\pi f_c^1 t) & \text{for information bit 1} \\ a\cos(2\pi f_c^0 t) & \text{for information bit 0} \end{cases}$$

Digital Modulation
FSK Modulation



The GSM Modulation

Remember FM modulation?

An FM modulated signal is given by

$$s_{\text{FM}}(t) = a_0 \cos\left(2\pi f_c t + 2\pi k_f \int_0^t x(\tau)d\tau\right)$$

where $x(t)$ is the information signal.

The GSM Modulation

GMSK Modulation

The GSM system uses Gaussian Minimum Shift Keying (GMSK) modulation, which is a *digital* version of FM

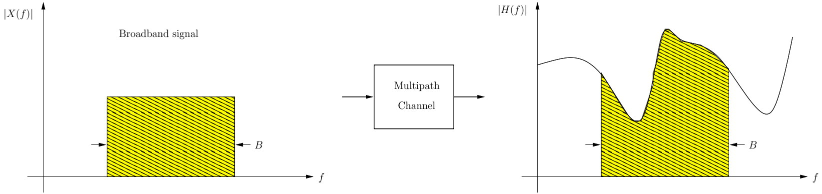
$$s_{\text{GMSK}}(t) = a_0 \cos\left(2\pi f_c t + 2\pi k_f \int_0^t x(\tau) d\tau\right)$$

- $x(t) = \sum_k a_k p(t - kT)$ is the digital information signal
- $p(t) = ce^{-\pi c^2 t^2}$ is the Gaussian pulse
- GMSK controls bandwidth expansion and has a compact spectrum

Multicarrier Modulation

Frequency Selective Fading Channels

Remember when $B \gtrapprox B_c$, the wireless channel introduces severe distortion to the transmitted signal. More formally, the channel is said to be frequency selective.

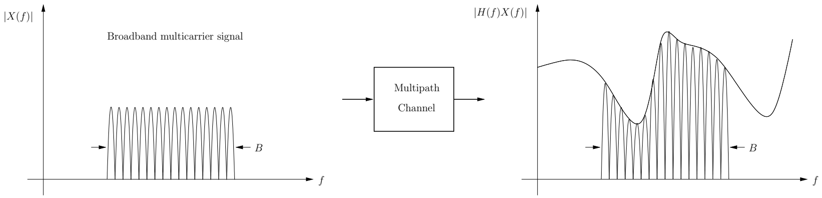


We need to equalise the channel, since different frequencies suffer different attenuations. We will now study what WiFi does...

Multicarrier Modulation

The Key Idea...

... is to modulate the signal with multiple carriers, so that the signal modulated onto each carrier looks narrowband and experiences a flat fading coefficient.



Multicarrier Modulation

How do we implement it?

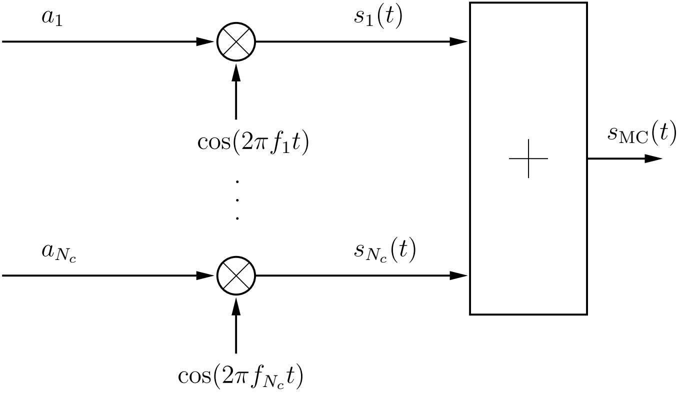
Consider that we have N_c subcarriers whose frequencies are $f_1, \ldots, f_{N_c}$ and that we divide our bits in N_c classes $a_1, \ldots, a_{N_c}$ to modulate each carrier. We can write the n-th modulated signal as

$$s_n(t) = a_n \cos 2\pi f_n t \quad 0 \leq t \leq T$$

and the multicarrier signal is

$$s_{\mathrm{MC}}(t) = \sum_{n=1}^{N_c} s_n(t) = \sum_{n=1}^{N_c} a_n \cos 2\pi f_n t \quad 0 \leq t \leq T$$

Multicarrier Modulation



Multicarrier Modulation

The choice of the sub-carriers

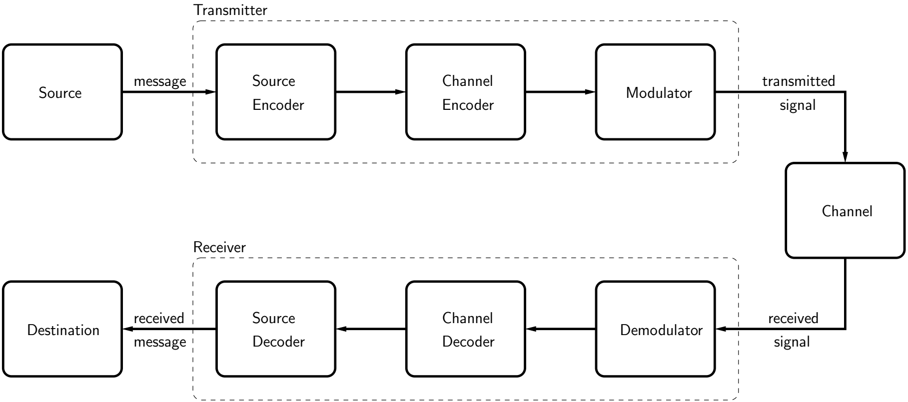
We choose the sub-carrier frequencies to be orthogonal, i.e.,

$$\int_0^T \cos(2\pi f_n t)\cos(2\pi f_m t) = 0 \qquad \forall m \neq n$$

- this can be achieved by choosing $f_n = nf_c$ and $f_m = mf_c$
- We can recover the data streams $a_1, \ldots, a_{N_c}$ if the sub-carriers are orthogonal
- We can have a much more compact spectrum than if we had non-orthogonal carriers
- Orthogonal Frequency Division Multiplexing (OFDM), used in ADSL, WiFi, and many future wireless systems

Basic Block Diagram

from Handout 1



Communications Channels

Definition

The medium used to transmit the signal from transmitter to receiver.

and...

introduces attenuation and noise so that the received signal is a faded and noisy version of what the transmitter sent.

which implies that...

noise and attenuation introduce errors.

Communications Channels

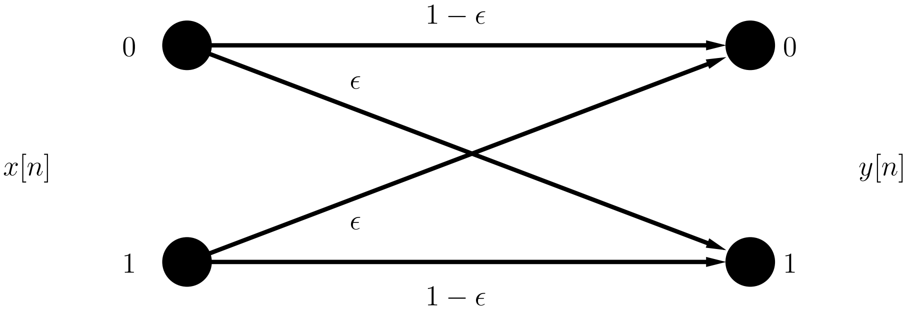
Error Probability

We define the bit error probability P_e (also bit error rate (BER)) as the probability that one bit is detected in error at the receiver.

- Fundamental measure of the reliability of transmission
- Tells us what the percentage of errors is
- P_e depends on the amount of noise in the channel
- Also measures the quality of service (QoS) requirements (acceptable quality) of a given system
  - $P_e \approx 10^{-3}$ for voice (GSM)
  - $P_e \approx 10^{-15}$ magnetic recording

An Illustrative Example

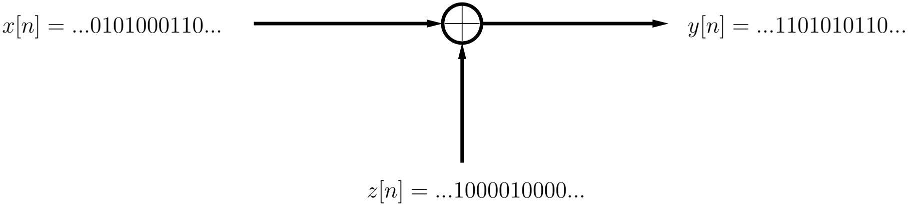
The Binary Symmetric Channel



An Illustrative Example

The Binary Symmetric Channel

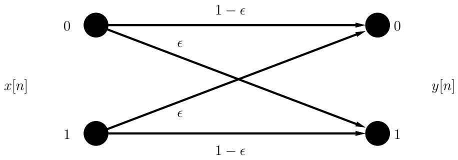
or equivalently...



- $x[n], y[n]$ transmitted and received binary sequences
- $z[n]$ additive binary noise sequence with $\Pr(z = 1) = \epsilon$, namely, with probability ϵ the channel introduces an error

An Illustrative Example

The Binary Symmetric Channel



Error Probability

The bit error probability is

$$P_e = \Pr(y=1|x=0)\Pr(x=0) + \Pr(y=0|x=1)\Pr(x=1)$$
$$= \epsilon\,\frac{1}{2} + \epsilon\,\frac{1}{2} = \epsilon$$

Can we improve it? How? Let's repeat the bits and see...

An Illustrative Example

The Binary Symmetric Channel

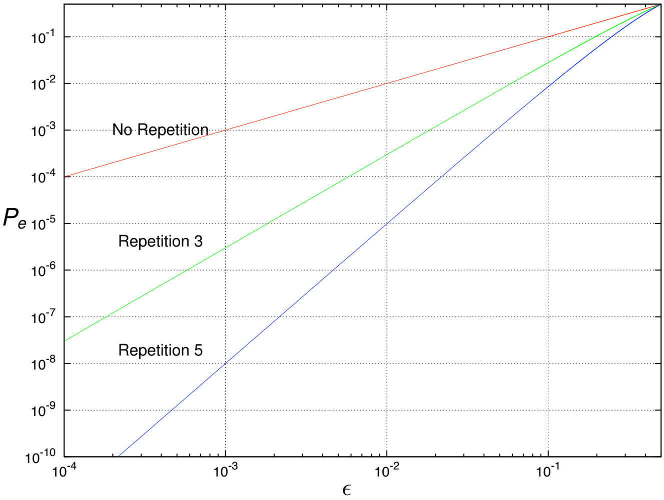
Repeat 3 Times

- Introducing redundancy: channel coding
- We transmit 1 information bit every 3 transmissions
- We need at least 2 out of 3 bits in error to declare an information bit error

$$P_e = p(3 \text{ errors}) + p(2 \text{ errors}) = \epsilon^3 + 3\epsilon^2(1-\epsilon)$$

An Illustrative Example

The Binary Symmetric Channel



An Illustrative Example

The Binary Symmetric Channel

Repetition Codes

- As we increase the repetition factor, we can achieve reliable communication, i.e., $P_e \longrightarrow 0$.
- Unfortunately $R \longrightarrow 0$ as well!!
- It was widely thought that in order to achieve $P_e \longrightarrow 0$ we had to have $R \longrightarrow 0$.
- Until 1948

Shannon's Theorem

Theorem

(Shannon 1948) For any communications channel, there exist codes (with the corresponding encoders and decoders) of rate $R < C$ *that have* $P_e \to 0$*, where* C *is the channel capacity. Conversely, if* $R > C$*,* P_e *cannot approach* 0*, and reliable transmission is not possible.*

Shannon's Theorem

The Binary Symmetric Channel

Capacity of the BSC

In the case of the binary symmetric channel

$$c = 1 - h(\epsilon) \quad \text{bits/channel use,}$$

where

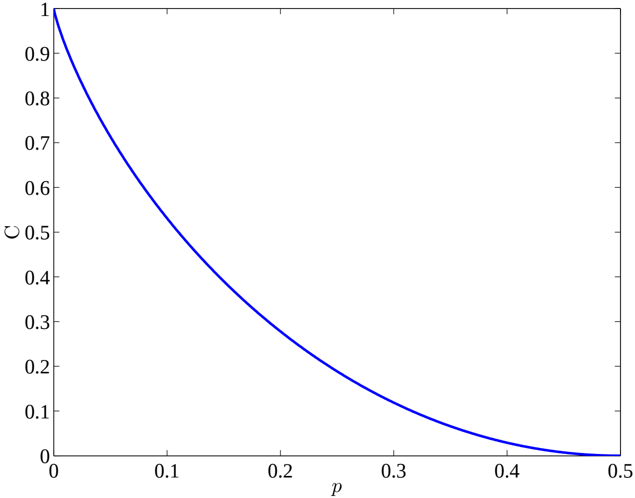
$$h(\epsilon) \triangleq \epsilon \log_2 \frac{1}{\epsilon} + (1 - \epsilon) \log_2 \frac{1}{1 - \epsilon}$$

is the binary entropy function.

- For $\epsilon = 0$ (noiseless channel) we can transmit up to $c = 1$ bits *every time we use the channel*
- For $\epsilon = \frac{1}{2}$, the channel generates a random output, so we guess, and $c = 0$ bits per transmission
- If every use of the channel corresponds to T seconds, then we define $C = \frac{c}{T}$ bit/second

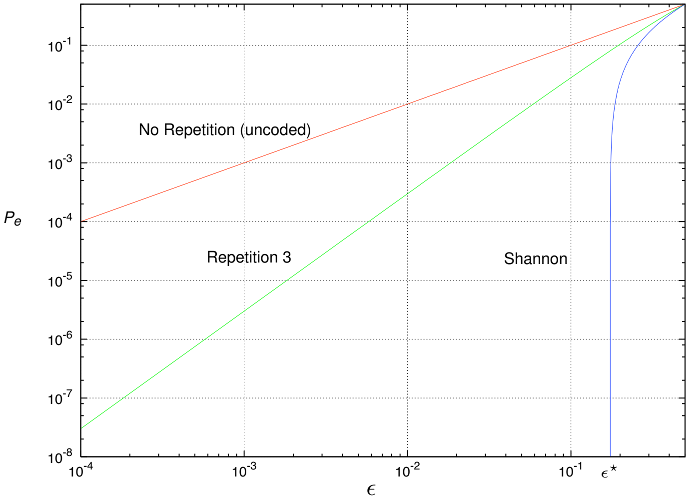
Shannon's Theorem

The Binary Symmetric Channel



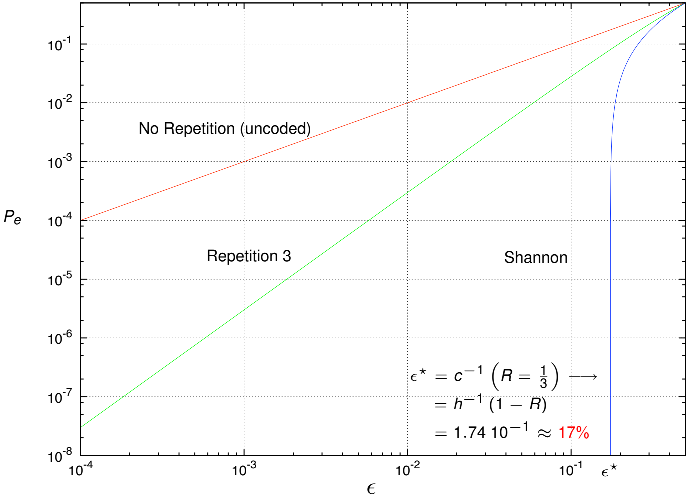
An Illustrative Example

The Binary Symmetric Channel



An Illustrative Example

The Binary Symmetric Channel



The Additive Gaussian Noise Channel

Capacity

The capacity (in bits/second) of the additive white Gaussian channel with bandwidth B is given by (Shannon 1948)

$$C = B\log_2\left(1 + \frac{P}{N_0 B}\right) = B\log_2(1 + \mathrm{SNR})$$

where $\mathrm{SNR} = \frac{P}{N} = \frac{P}{N_0 B}$. This formula is in the data book.

- Optimally relates our *precious* resources: P, B and R
- The spectral efficiency $c = \frac{C}{B} = \log_2(1 + \mathrm{SNR})$ is the normalized (wrt B) capacity in bits per second per Hertz
- A linear increase in power (or SNR) implies logarithmic increase in rate.

The Additive Gaussian Noise Channel

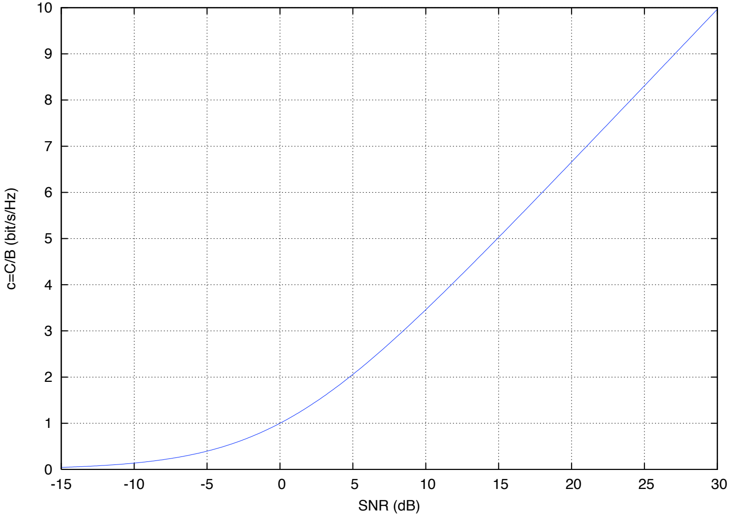
Recap...

With sufficiently involved coding techniques, we can transmit binary digits at a rate

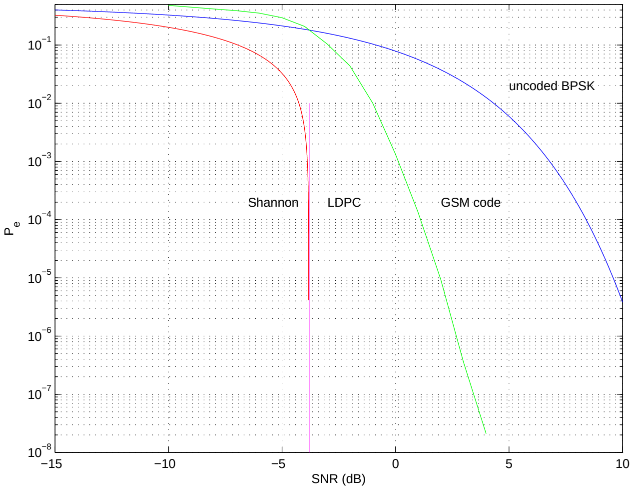
$$R < B \log_2 (1 + \mathrm{SNR})$$

bits per second, with arbitrarily small probability of decoding in error, namely $P_e \rightarrow 0$. Conversely, if $R > C$ reliable communication is not possible since P_e cannot vanish.

The Additive Gaussian Noise Channel



The Additive Gaussian Noise Channel



$$\mathrm{SNR}^\star = c^{-1}\left(R = \tfrac{1}{2}\right) = 2^R - 1 = 0.41 \longleftrightarrow N^\star = 2.41\,P^\star$$