IB Paper 6: Signal and Data Analysis

Handout 2: Fourier Series

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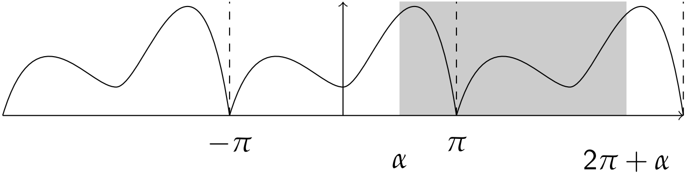
Fourier Series – Revision of Basics



Real Fourier Series

Any function, $g(t)$, which is **periodic** in the interval $[-\pi, \pi]$ has a *real Fourier Series* representation given by

$$g(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\{a_n \cos(nt) + b_n \sin(nt)\} \tag{1}$$

The coefficients are given by:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(t)\cos(nt)dt \qquad n = 0, 1, ..., \infty \tag{2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(t)\sin(nt)dt \qquad n = 1, 2, ..., \infty \tag{3}$$

i.e. Any periodic function can be formed from a linear combination of the functions $\cos(nt)$ and $\sin(nt)$.

Complex Fourier Series

An equivalent *complex* Fourier series representation is given by

$$g(t) = \sum_{n=-\infty}^{\infty} c_n e^{jnt} \tag{4}$$

with coefficients:

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) e^{-jnt} dt \quad n = -\infty, ..., -1, 0, 1, ..., +\infty \tag{5}$$

We are effectively **decomposing the function $g(t)$ into its frequency components** - each e^{jnt} is a complex oscillating function with frequency n rad.s^{-1}. We will focus from now on complex Fourier series, as they are the basis for Fourier Transforms and Discrete Fourier Transforms in later parts of the course.

Proof of expression for coefficients

You should be familiar with how to prove the above expressions for coefficients from 1a. The method of proof is important though for development of Fourier Transforms later on, so we reiterate it here. Take the complex case, for example. Multiply $g(t)$ in equation (**??**) by e^{-jmt} and integrate wrt t from $-\pi$ to π:

$$
\begin{aligned}
I &= \int_{-\pi}^{\pi} g(t) e^{-jmt} dt \\
&= \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} c_n e^{j(n-m)t} dt && \text{[From Eq.(4)]} \\
&= \sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} c_n e^{j(n-m)t} dt && \text{[Swap order of int. and summ.]} \\
&= \int_{-\pi}^{\pi} c_m dt + \sum_{\substack{n=-\infty \\ n \neq m}}^{\infty} \int_{-\pi}^{\pi} c_n e^{j(n-m)t} dt && \text{[Separate out term } n = m\text{]}
\end{aligned}
$$

$$
\begin{aligned}
&= 2\pi c_m + \sum_{\substack{n=-\infty \\ n\neq m}}^{\infty} c_n \int_{-\pi}^{\pi} e^{j(n-m)t} dt \\
&= 2\pi c_m + \sum_{\substack{n=-\infty \\ n\neq m}}^{\infty} c_n \int_{-\pi}^{\pi} \cos((n-m)t) + j\sin((n-m)t) dt \\
&= 2\pi c_m \qquad (6)
\end{aligned}
$$

since

$$\int_{-\pi}^{\pi} \cos(kt) dt = \int_{-\pi}^{\pi} \sin(kt) dt = 0$$

when k is an integer not equal to 0.

We therefore have an expression for the complex Fourier coefficients c_n;

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) \mathrm{e}^{-jnt} dt. \tag{7}$$

You can prove the formulae for a_n and b_n by a similar procedure, but multiplying by $\cos(mt)$ and $\sin(mt)$ instead of $\exp(imt)$.

Relationship between real and complex coefficients

By splitting Eq. (**??**) into real and imaginary parts we see that

$$2c_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(t)\cos(nt)dt - j\frac{1}{\pi}\int_{-\pi}^{\pi} g(t)\sin(nt)dt = a_n - jb_n$$

for $n \geq 0$ (with $b_0 = 0$).We therefore have the relationship between the complex and real Fourier coefficients (for $g(t)$ real):

$$2c_n = a_n - jb_n\ , \qquad 2c_n^* = a_n + jb_n \quad \text{for} \quad n \geq 0 \tag{8}$$

and conversely by solving for a_n and b_n:

$$a_n = c_n^* + c_n\ , \qquad jb_n = c_n^* - c_n \quad \text{for} \quad n \geq 0 \tag{9}$$

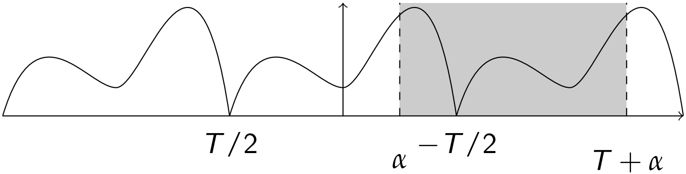
From Eq. (**??**) we can see that $c_{-n} = c_n^*$ and therefore we have from Eq. (**??**)

$$2c_{-n} = a_n + jb_n\ , \qquad 2c_{-n}^* = a_n - jb_n \quad \text{for} \quad n > 0$$

Finally the modulus of c_n is easily obtained in terms of the real coefficients:

$$2|c_n| = \sqrt{a_n^2 + b_n^2}$$

Signals with period T



When a periodic signal has period T rather than 2π, simply think of 'stretching' the time axis by a factor of $T/2\pi$. The complex Fourier series, for example, becomes:

$$g(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_o t}. \tag{10}$$

where $\omega_o = \frac{2\pi}{T}$ - known as the 'fundamental frequency'.

To obtain the formula for the coefficients, substitute $2\pi t/T = \omega_0 t$ for t into the coefficient formulae:

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} g(t)\mathrm{e}^{-j\omega_o nt}dt \qquad (11)$$

As usual, we can redefine the range of integration to be any whole period of $g(t)$, e.g. α to $\alpha + T$:

$$c_n = \frac{1}{T}\int_{\alpha}^{\alpha+T} g(t)\mathrm{e}^{-j\omega_o nt}dt \qquad (12)$$

Further Properties of Fourier Series

Scaling, stretching and shifting

Now consider modifying $g(t)$ by scaling the amplitude by a factor of a, shifting it along the axis by b and changing the period to $T' = \beta T$. What is the modified Fourier series? We can write the modified signal as:

$$g'(t) = ag\left(\frac{t-b}{\beta}\right) \tag{13}$$

The Fourier series is hence

$$
\begin{aligned}
g'(t) = ag\left(\frac{t-b}{\beta}\right) &= a\sum_{n=-\infty}^{\infty} c_n e^{j\omega_o n\frac{(t-b)}{\beta}} \\
&= \sum_{n=-\infty}^{\infty} \left\{ ac_n e^{-\frac{j\omega_o nb}{\beta}} \right\} e^{\frac{j\omega_o nt}{\beta}} \\
&= \sum_{n=-\infty}^{\infty} c_n' e^{j\omega_o' nt}
\end{aligned}
$$

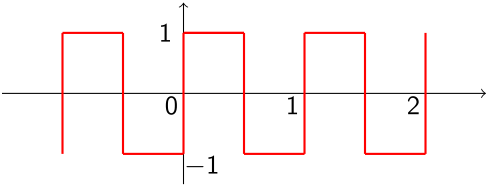
where the new coefficients and fundamental frequency are:

$$
c_n' = ac_n e^{-j\omega_o' nb} \qquad \omega_0' = \omega_0/\beta \tag{14}
$$

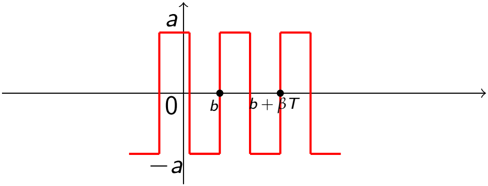
Note the effects of these actions on the frequency components.

Example: Square Wave:

Consider the square wave periodic function, amplitude 1, period $T = 1$;



Write down the Fourier series of this square wave and using this, obtain the Fourier series of the modified square wave having amplitude a, period $T' = \beta T$, and shifted along the time axis by b:



The original periodic square wave can be written as the following Fourier series (see Data Book)

$$g(t) = \sum_{\substack{n=-\infty \\ n \text{ odd}}}^{\infty} \frac{2}{j\pi n} e^{j\omega_o n t} \qquad \omega_o = \frac{2\pi}{T}.$$

Using our previous result gives directly the new Fourier coefficients:

$$c'_n = \frac{2a}{j\pi n} e^{-j\omega'_0 n b} \quad \text{for } n \text{ odd}$$

where $\omega'_0 = \omega_0/\beta$.

Differentiation and Integration

If we know the Fourier series representation of a function $g(t)$ then it is relatively straightforward to find the Fourier series for $\frac{dg(t)}{dt}$ and $\int g(t)dt$.

Suppose

$$g(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_o t}. \tag{15}$$

Then, differentiating gives us

$$\frac{dg(t)}{dt} = \sum_{n=-\infty}^{\infty} (jn\omega_o c_n) e^{jn\omega_o t} = \sum_{n=-\infty}^{\infty} c_n' e^{jn\omega_o t}$$

where

$$c_n' = jn\omega_o c_n. \tag{16}$$

Now, suppose $\frac{dh(t)}{dt} = g(t)$. Integrate $g(t)$ directly to find $h(t)$:

$$h(t) = \int g(t)dt = c_0 t + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{c_n}{(jn\omega_o)} e^{jn\omega_o t} + k \qquad (17)$$

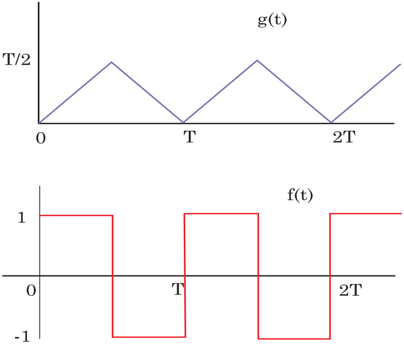
We can see that this contains a linear ramp term $c_0 t$, and so the integral can only be a Fourier series when $c_0 = 0$, i.e. $g(t)$ has zero mean.

To determine the constant of integration k, use standard Fourier series formula for $h(t)$ and $n = 0$. Thus the coefficients d_n for $h(t)$ are:

$$d_n = \begin{cases} \frac{c_n}{jn\omega_o} & \text{if } n \neq 0 \\ \frac{1}{T}\int_0^T h(t)dt & \text{if } n = 0 \end{cases}$$

Example

Consider the following waveform, $g(t)$, with height $T/2$ and period T



We wish to find the Fourier series expansion for $g(t)$ via use of one of the standard series on the data sheet.

We look to see if the signal is related to one of our standard signals by differentiation or integration; in this case it is clear that $\frac{dg(t)}{dt} = f(t)$, where $f(t)$ is the square wave of amplitude 1 and period T, as shown in the figure.
We know from the data sheet that the Fourier series for $f(t)$ is

$$f(t) = \sum_{\substack{n=-\infty \\ n \text{ odd}}}^{\infty} \frac{2}{j\pi n} e^{jn\omega_o t}$$

$(\omega_0 = 2\pi/T)$. From the above we have the Fourier series for $g(t)$:

$$c_n = \begin{cases} \frac{2}{j\pi n}\frac{1}{jn\omega_o} = -\frac{T}{\pi^2 n^2} & n \neq 0, \ n \text{ odd} \\ \frac{1}{T}\int_0^T g(t)dt = \frac{T}{4} & n = 0 \end{cases}$$

Can check this from series for triangular wave (need to shift, scale etc.) in the data book. See now Ex. Paper 6-6 Qqs. 1-2

Interpretation of Fourier Coefficients

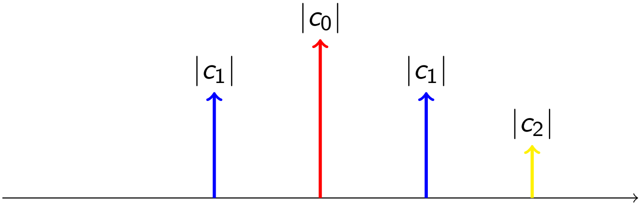
Consider the complex Fourier series for a periodic signal $g(t)$:

$$g(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_o t}. \tag{18}$$

Each complex exponential $e^{jn\omega_o t}$ can be regarded as a pure *frequency* component of the signal - an oscillating signal containing only sines and cosines with frequency $n\omega_o t$.

The component with frequency ω_o is known as the *fundamental frequency*, or *first harmonic*, $2\omega_o$ is the second harmonic, ..., $n\omega_o$ is the *n*th harmonic.

We can represent this frequency content graphically:



The amplitudes r_0, r_1, ...,r_n of the harmonics are defined by rewriting the real Fourier series for $g(t)$ in terms of a d.c. term plus pure sine waves:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \{a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)\} = r_0 + \sum_{n=1}^{\infty} r_n \sin(n\omega_0 t + \phi_n)$$

so that $r_0 = a_0/2$ and $r_n = \sqrt{(a_n^2 + b_n^2)}$ by standard trigonometric identities. From equation (**??**) we can therefore express this amplitude in terms of the c_n's, for $n \geq 1$:

$$r_n \;=\; \sqrt{(a_n^2 + b_n^2)} = 2|c_n| \equiv |c_{-n}| + |c_n|.$$

[i.e. r_n contains both the negative and positive frequency components of c_n]

and, for $n = 0$, the DC component,

$$r_0 = a_0/2 = c_0$$

You need this result for Ex. Paper 6/6 Q.3 !.

These amplitudes can be related to the power content of the signal $g(t)$ over one period (or the average energy over one period). The following expression holds

$$\frac{1}{T}\int_0^T |g(t)|^2 dt = \sum_{n=-\infty}^{\infty} |c_n|^2 \tag{19}$$

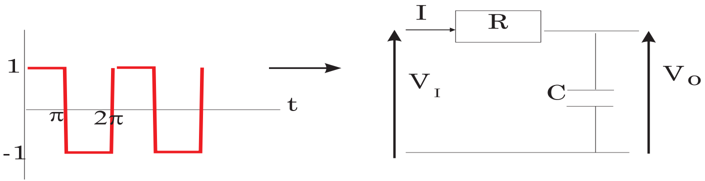
This is **Parseval's Theorem** for Fourier series (we will come across Parseval's Theorem again later)

.... you will show a more general version of this on the example sheet 6/6 Q.7 .

Example

As a final example dealing with Fourier series we will look at passing a periodic signal through a filter and interpreting the effect of the filter by its action on the harmonics.

Consider a square wave of period $T = 2\pi$ and amplitude 1 which is fed into an RC circuit given below



Question: *Determine the amplitudes of the harmonics of the output waveform v_o.*

Plan of Solution:

Summary of approach:

1. Obtain Fourier series for $V_i \rightarrow c_n$.
2. Find frequency response of RC-circuit $H(j\omega) = H(s)|_{s=j\omega}$
3. Determine amplitude of nth harmonic at output as $2|c_n| \times |H(jn\omega_o)|$

Detailed solution:

1. Find Fourier expansion of input

From the EI data book we know that the Fourier series expansion for the input square wave, V_i, is

$$V_i = \sum_{\substack{n=-\infty \\ n \text{ odd}}}^{\infty} \frac{2}{j\pi n} e^{jn\omega_o t}, \qquad \omega_o = \frac{2\pi}{T} = 1$$

Using the definition of harmonic amplitudes, the amplitude of the nth harmonic is thus:

$$r_n = 2|c_n| = \frac{4}{n\pi}, \quad n = 1, 3, 5, \ldots$$

The d.c. amplitude r_0 is zero. Since the square wave is an *odd* function we have:

$$g(t) = \sum_{n=1,3,5,\ldots}^{\infty} \frac{4}{n\pi} \sin(n\omega_0 t)$$

2. Find frequency response of the RC-circuit

Since $I = (V_i - V_o)/R = C\frac{dV_0}{dt}$ we can take Laplace transforms to give

$$\overline{V}_i - \overline{V}_o = RCs\overline{V}_o \implies \frac{\overline{V}_o}{\overline{V}_i} = \frac{1}{1+sRC}.$$

We replace s by $j\omega$ to obtain the frequency response:

$$H(j\omega) = \frac{1}{1+j\omega RC}$$

Recall also that if the input signal to a linear system is a sine wave

$$\sin(\omega t + \theta)$$

and the frequency response is $H(j\omega) \equiv |H(j\omega)| \exp(j \arg(H(j\omega)))$, then the output signal (after initial transients have died out) is,

$$|H(j\omega)| \sin(\omega t + \theta + \arg(H(j\omega)))$$

3. Find effect of RC circuit on amplitude of nth harmonic

$$|H(jn\omega_0)| = \left|\frac{1}{1 + jn\omega_0 RC}\right| = \frac{1}{\sqrt{(1 + n^2\omega_0^2(RC)^2)}}$$

The nth harmonic is therefore subject to a decrease in amplitude with increasing n. As RC increases, the amplitudes of the harmonics are more attenuated and the waveform has the characteristic exponential rise-and-fall form expected of this circuit ('low-pass filter').

This whole process is illustrated in figure **??**

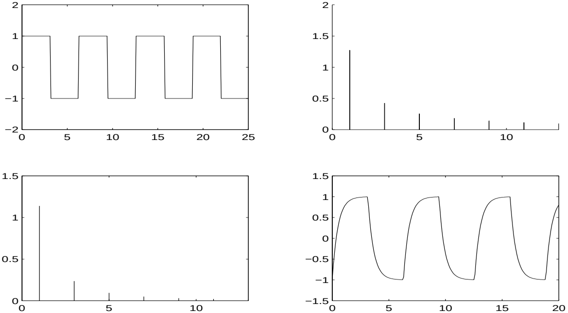


Figure 1 : a) Original square wave input, b) frequency content of the square wave, c) frequency content modified by RC circuit, d) output signal. Plotted for RC=0.5

See now Ex. Paper 6/6 Q.3