IB Paper 6: Signal and Data Analysis

Handout 4: Properties of the Fourier Transform

S Godsill

Signal Processing and Communications Laboratory,
Engineering Department,
Cambridge, UK

Lent 2015

Properties of the Fourier Transform

We recall that if we knew the Fourier coefficients for some standard functions we were able to find the Fourier coefficients for other functions related to the original by *shifts, scalings, differentiation, etc.*. We now want to look at the corresponding properties of the Fourier transform

See also EI Data book for a fairly complete list of properties.

Linearity

The Fourier Transform is linear, just like the Laplace transform and Fourier series. For a linear function the following result applies:

$$f(t) = af_1(t) + bf_2(t) \xleftrightarrow{FT} aF_1(\omega) + bF_2(\omega) = F(\omega) \qquad (1)$$

where a and b are scalar constants. This is really just saying that integration is a linear operator - can check this result for yourself. This means we can find the Fourier transform of complicated functions by decomposing them as a sum of simpler functions ($f(t) = af_1(t) + bf_2(t)$) and then summing the transforms of the simpler functions ($F(\omega) = aF_1(\omega) + bF_2(\omega)$).

Time Scaling / Similarity Theorem

If $f(t)$ has Fourier transform $F(\omega)$, what is the Fourier transform of $f(\alpha t)$, the time-scaled version of $f(t)$?
Start with the definiton of the inverse FT:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

Now simply substitute αt for t:

$$f(\alpha t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega(\alpha t)} d\omega$$

For $\alpha > 0$, substitute $\omega' = \alpha\omega$, hence $d\omega = (1/\alpha) d\omega'$ and limits are unchanged:

$$f(\alpha t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \frac{1}{\alpha} F\left(\frac{\omega'}{\alpha}\right) \right\} e^{j\omega' t} d\omega'$$

Studying the form of this equation, we can see that it is precisely in the form of an inverse Fourier transform, where the frequency function is $\frac{1}{\alpha} F\left(\frac{\omega'}{\alpha}\right)$. Hence:

$$\boxed{f(\alpha t) \overset{FT}{\longleftrightarrow} \frac{1}{\alpha} F\left(\frac{\omega}{\alpha}\right)} \tag{2}$$

Thus, if we stretch in the time domain we contract in the frequency domain and vice versa.

(This results also holds for $\alpha < 0$ – you can check this is the case).

Example

Recall the example of the rectangular pulse near the end of Handout 3.
The formula for the pulse is given by

$$f(t) = \begin{cases} b & \text{for } -T/2 < t < T/2 \\ 0 & \text{otherwise} \end{cases}$$

and has Fourier transform:

$$bT\text{sinc}\left(\frac{\omega T}{2}\right)$$

Thus $f(2t)$, which is a pulse of width $T/2$, will have FT

$$\frac{bT}{2}\text{sinc}\left(\frac{\omega T}{4}\right)$$

thus producing a spectrum of double the width.

Heisenberg-Gabor principle

We note that increasing α in the time-stretched function makes $f(\alpha t)$ proportionally narrower.

However, as α increases, $\frac{1}{\alpha}F\left(\frac{\omega}{\alpha}\right)$ becomes proportionally 'wider', and *vice versa*:

In fact there is a general principle, the *Heisenberg-Gabor* principle, which formalises this idea:

> *If any function $f(t)$ has time duration T, and its Fourier transform $F(\omega)$ has frequency bandwidth B, then,*
>
> $TB \geq 1$ *(Time-Bandwidth product)*

This can be proved quite elegantly, but we won't go into the details here.

Time Shift

If $f(t)$ has FT $F(\omega)$, what is the FT of $f(t-t_0)$ (time shifted by $+t_0$)? Again, start with the inverse FT formula:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega$$

Substitute $t-t_0$ for t to give

$$\begin{aligned} f(t-t_0) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega(t-t_0)}d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left\{F(\omega)e^{-j\omega t_0}\right\} e^{j\omega t}d\omega \end{aligned}$$

Again, the integral is itself in the form of an inverse FT, so that:

$$f(t-t_0) \xleftrightarrow{FT} F(\omega)e^{-j\omega t_0} \qquad (3)$$

You get to prove this for yourself - see examples paper 6/6 question 4.

Frequency Shift or Modulation

If $f(t)$ has Fourier transform $F(\omega)$, what function has Fourier transform $F(\omega - \omega_0)$?
This time, start with forward transform formula and substitute $\omega - \omega_0$ for ω:

$$F(\omega - \omega_0) = \int_{-\infty}^{\infty} f(t) e^{-j(\omega - \omega_0)t} dt = \int_{-\infty}^{\infty} \{f(t) e^{j\omega_0 t}\} e^{-j\omega t} dt$$

so that

$$e^{j\omega_0 t} f(t) \xleftrightarrow{FT} F(\omega - \omega_0) \tag{4}$$

This result is fundamental to the **modulation** of signals (see Communications course - 2nd half of Lent Paper 6).

From the above modulation theorem we can deduce the FT of $f(t)\cos(\omega_0 t)$ in terms of the FT of $f(t)$, since

$$f(t)\cos(\omega_0 t) = \frac{1}{2}\{e^{j\omega_0 t}f(t) + e^{-j\omega_0 t}f(t)\}$$

Hence, by linearity of the FT:

$$f(t)\cos(\omega_0 t) \overset{FT}{\longleftrightarrow} \tfrac{1}{2}F(\omega - \omega_0) + \tfrac{1}{2}F(\omega + \omega_0) \qquad (5)$$

The main applications of this occur in mobile comms., radio and television, where a harmonic **carrier wave** ($\cos \omega_0 t$ in this case) is multiplied ('modulated') by an envelope ($f(t)$ - containing the broadcast signal).

The result is that the spectrum of the envelope is separated into two parts, each half of its original strength, and shifted along the ω axis by $\pm\omega_0$.

Differentiation wrt t

If $f(t)$ has Fourier transform $F(\omega)$, what is the Fourier transform of $\frac{df}{dt}$?

$$\begin{aligned} f'(t) &= \frac{d}{dt}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\frac{d}{dt}\left\{e^{j\omega t}\right\}d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\{j\omega F(\omega)\}\, e^{j\omega t}d\omega \end{aligned}$$

which therefore tells us that

$$f'(t) \xleftrightarrow{FT} j\omega F(\omega) \tag{6}$$

Then, it is clear that repeated differentiation will simply bring down further factors of $(j\omega)$ – which leads to the result

$$f^{(n)}(t) \xleftrightarrow{FT} (j\omega)^n F(\omega) \tag{7}$$

where $f^{(n)}(t)$ denotes the nth derivative of $f(t)$

Duality

Suppose the Fourier transform of $f(t)$ is $g(\omega)$. What is the Fourier transform of $g(t)$ (i.e. the same function $g(\omega)$ interpreted as a function of time)?
Inverse FT gives:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\omega)e^{j\omega t}d\omega$$

Now, if we replace t in the above by $-\omega'$ and rename ω with t' (ω is simply an integration variable), we have

$$f(-\omega') = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(t')e^{-j\omega' t'}dt'$$

Rearranging this gives:

$$2\pi f(-\omega') = \int_{-\infty}^{\infty} g(t')e^{-j\omega' t'}dt'$$

where the RHS is exactly the FT of $g(t')$.

This is a labour-saving result. If we have one Fourier transform pair:

$$f(t) \stackrel{FT}{\longleftrightarrow} g(\omega)$$

then we automatically have (without any integration) the *dual* Fourier transform pair:

$$g(t) \stackrel{FT}{\longleftrightarrow} 2\pi f(-\omega)$$

This can be difficult to understand at first read-through. See also Example Sheet 6/6, Q4.

Example:

Saw previously (and it is in the databook) that the Fourier transform of the rectangular pulse centred on the origin (width T, height b) was

$$bT\mathrm{sinc}(\omega T/2).$$

Hence, by duality, the Fourier transform of $bT\mathrm{sinc}(tT/2)$ is a rectangle pulse with height $2b\pi$ and width T

The Multiplication Theorem and Parseval's Theorem

Consider two functions $f_1(t)$ and $f_2(t)$ with FTs $F_1(\omega)$ and $F_2(\omega)$ and look at the integral of the product of f_1 and f_2^*:

$$\begin{aligned}\int_{-\infty}^{\infty} f_1(t) f_2^*(t)\,dt &= \int_{-\infty}^{\infty} f_1(t)\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} F_2^*(\omega)\mathrm{e}^{-j\omega t}d\omega\right\}dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F_2^*(\omega)\left\{\int_{-\infty}^{\infty} f_1(t)\mathrm{e}^{-j\omega t}dt\right\}d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(\omega)F_2^*(\omega)\,d\omega\end{aligned}$$

Putting $f_1(t) = f_2(t)$ in the above leads to **Parseval's theorem**

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega \tag{8}$$

Note the similarity between this and Parseval's theorem for Fourier series – here it is telling us that the amount of 'energy' in a system can be found either by integrating in the time domain or (equivalently) in the spectral domain.

Parseval's Theorem: Example

Here we find the energy of the sinc function, $f(t) = \frac{sin(t)}{t}$:

$$E = \int_{-\infty}^{\infty} \left| \frac{\sin(t)}{t} \right|^2 dt$$

This is a hard integral directly in the time domain. However, we do know the Fourier Transform of the sinc function as a rectangle pulse:

$$F(\omega) = \begin{cases} \pi, & -1 < \omega < +1 \\ 0, & \text{otherwise} \end{cases}$$

Thus applying Parseval directly, we have:

$$E = \int_{-\infty}^{\infty} \left| \frac{\sin(t)}{t} \right|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega = \frac{\pi^2 \times 2}{2\pi} = \pi$$

A hard integral has thus been turned into a simple one over a rectangle function in the frequency domain.

We can apply a simple extension to work out questions like 'How much energy of the sinc function lies between frequencies ω_1 and ω_2?'
The answer is then:

$$\frac{1}{2\pi}\int_{-\omega_2}^{-\omega_1}|F(\omega)|^2 d\omega + \frac{1}{2\pi}\int_{\omega_1}^{\omega_2}|F(\omega)|^2 d\omega = 2\times\frac{1}{2\pi}\int_{\omega_1}^{\omega_2}|F(\omega)|^2 d\omega$$

[last equality applies only for real-valued signals].
Note that we have to include both the positive and negative frequency ranges in the integral to get the solution here.
Note that we can express Parseval in terms of Hz frequency via the substitution $\omega = 2\pi f$, $d\omega = 2\pi df$:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}|F(\omega)|^2 d\omega = \int_{-\infty}^{\infty}|F(2\pi f)|^2 df$$

See now Examples Paper 6 qqs. 7-9

Convolution

The convolution of two functions $f(t)$ and $g(t)$ is written as $h(t) = f * g$ and defined by

$$h(t) = f * g = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau. \tag{9}$$

The **Convolution Theorem** for Fourier transforms is as follows. Suppose $f(t)$ and $g(t)$ have FTs $F(\omega)$ and $G(\omega)$. Then,

$$\text{If } h(t) = f * g \implies H(\omega) = F(\omega)G(\omega). \tag{10}$$

cf. similar result for Laplace Transforms.

We now show this. If $h(t) = f * g$, taking the FT of the convolution gives

$$H(\omega) = \int_{t=-\infty}^{\infty} \left\{ \int_{\tau=-\infty}^{\infty} f(\tau) g(t-\tau)\, d\tau \right\} \mathrm{e}^{-j\omega t}\, dt. \qquad (11)$$

Change the order of integration and substitute $u = t - \tau$, [$\Longrightarrow$ $dt = du$ and no change of limits]:

$$\begin{aligned} H(\omega) &= \int_{-\infty}^{\infty} f(\tau) \left[\int_{u=-\infty}^{\infty} g(u) \mathrm{e}^{-j\omega(u+\tau)} du \right] d\tau \\ &= \left\{ \int_{-\infty}^{\infty} f(\tau) \mathrm{e}^{-j\omega\tau} d\tau \right\} \left\{ \int_{-\infty}^{\infty} g(u) \mathrm{e}^{-j\omega u} du \right\} \\ &= F(\omega) G(\omega). \end{aligned}$$

Also, by duality (check this this for yourself):

$$\boxed{2\pi f(t)g(t) \xleftrightarrow{FT} F(\omega) * G(\omega)} \tag{12}$$

[This result is actually quite difficult to show by duality - be careful to use the definitions of $F()$ and $G()$ correctly].

Therefore we have the **VERY IMPORTANT** result that convolution in one domain is equivalent to multiplication in the other domain.

Fourier Transforms and Linear Systems

As with Laplace transforms, the convolution result for Fourier transforms can be used to analyse the effects of a system on an input signal.

Have seen examples of convolution with Laplace transforms in Linear Systems – we can do the same thing with Fourier transforms.

Let $h(t)$ be the response of a linear system to an impulse $\delta(t)$ (the 'impulse response').

Consider an input to this system of $x(t)$. Since it is a linear system the output, $y(t)$, is the convolution $x(t) * h(t)$:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau. \qquad (13)$$

Therefore from the convolution theorem, the FT of the output is the product of the FTs of x and h

$$Y(\omega) = X(\omega)H(\omega) \qquad (14)$$

which is the analogue of the result from Laplace transforms – the transfer function $H(\omega)$ is the FT of the impulse response and gives the ratio of the FT of the output to the FT of the input.

Fourier transform vs. Laplace transform

Compare the definitions:

$$F(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-j\omega t}\, dt \qquad \bar{f}(s) = \int_{0}^{+\infty} f(t) e^{-st}\, dt$$

Similarities: the integrand is the same if we set $s = j\omega$

Differences: the limits of integration are different (although note that sometimes you will see a bilateral Laplace Transform defined which has limits of $(-\infty, +\infty)$).

However, for a function $f(t)$ that is zero for $t < 0$, we have that

$$F(\omega) = \bar{f}(j\omega)$$

and the two are equivalent (provided both integrals exist).

In particular, for a *causal* linear time-invariant system, we can calculate the frequency response in either way, as follows.

Suppose the system has impulse response $h(t)$ with the following properties:

$$h(t) = 0, \qquad \text{for any } t < 0 \quad \text{(causal system)}$$

$$h(t) \overset{LT}{\leftrightarrow} \overline{h}(s) \quad \text{(Laplace transform)}$$

$$h(t) \overset{FT}{\leftrightarrow} H(\omega) \quad \text{(Fourier transform)}$$

Then the frequency response of the system can be calculated using either Fourier transforms or Laplace transforms since

$$H(\omega) = \overline{h}(j\omega) = \text{Frequency response}$$

.

Differential equations can often be solved using either Laplace transforms or Fourier transforms

- Laplace is better suited to problems with boundary conditions at $t = 0$,
- Fourier better suited to steady state analysis.

Note that standard 1A ac circuit theory is a simple application of Fourier transforms and frequency response to a differential equation system.

S. Godsill (2015), J. Lasenby (2009)