IB Paper 6: Signal and Data Analysis

Handout 5: Sampling Theory

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Sampling and Aliasing

- All of our work so far with Fourier series and Fourier transforms has worked with signals and functions in continuous time.

- Calculation of Fourier coefficients requires integrals over continuous time.

- This is fine when you consider special functions whose formula can be written down (sine, cosine, δ-function, etc.)
• However, in the real world, signals don’t have a pre-specified formula - we just have to measure them.

• Nowadays, signals are measured in digital form on computers, which means discrete time sampling, or analogue to digital conversion.

• Can we still do signal analysis when continuous time signals have been sampled and stored in digital format?

• The theory of sampling and aliasing shows how to do this in a proper fashion.
Digital Sampling

Figure 1: Digital sampling of a continuous waveform
Firstly define what is meant by **Digital sampling**:

- Suppose a continuous time signal is given by \( f(t) \), \(-\infty < t < +\infty\)

- Choose a **sampling interval** \( T \) and read off the value of \( f(t) \) at times:
  \[-\infty, ... -2T, -T, 0, T, 2T, ..., +\infty\]
  i.e. at times \( nT, n = -\infty, ..., -1, 0, 1, 2, ..., \infty \).

- The obtained values \( f(nT) \) are the **sampled** version of \( f(t) \).

- The practical procedure, known as **analogue to digital conversion**, is discussed further in the Communications course (P6 2nd half of Lent term).
The big question is: how should you choose $T$, the sampling interval?

- Too large a value of $T$ will mean loss of detail from $f(t)$:

![Sparse sampling of a continuous waveform](image)

**Figure 2**: Sparse sampling of a continuous waveform
• Too small a value means unnecessary storage of over-detailed (redundant) data:

![Dense sampling of a continuous waveform](image)

Figure 3: Dense sampling of a continuous waveform
• We need the ‘Goldilocks’ principle!
  ‘Not too big; and not too small; but just right’

But, seriously:

• The **Sampling Theorem** tells us the maximum value of $T$ we can take and still perfectly reconstruct $f(t)$ from $f(nT)$ - a remarkable and perhaps not obvious result.
The Sampling Theorem

Figure 4: Representing the sampled data as a train of $\delta$-functions
Firstly define a mathematical representation of the sampled signal using a train of $\delta$-functions:

Do this by taking each sample $f(nT)$ and multiplying it by a $\delta$-function centered at $t = nT$:

$$f(nT)\delta(t - nT)$$

But this equals

$$f(t)\delta(t - nT) \quad [\text{since } \delta(t - nT) \text{ is zero except at } t = nT]$$

Then sum all such samples to give the whole sampled signal as:

$$f_s(t) = \sum_{n=-\infty}^{\infty} f(t)\delta(t - nT)$$

$$= f(t) \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$= f(t) \delta_p(t)$$
\( f_s(t) \) is then a continuous time signal which contains only the sampled data information \( f(nT) \) - it is zero elsewhere.

**Note**: think of this as an **conceptual** version of the sampled signal – in no way are we implying that there are infinities in a real sampled signal.

![Train of delta functions, \( \delta_p(t) \)](image)

**Figure 5**: Train of delta functions, \( \delta_p(t) \)
\( \delta_p(t) \) is a periodic function and can therefore be represented as a Fourier series:

\[
\delta_p(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} \quad (1)
\]

where \( \omega_0 = \frac{2\pi}{T} \) and the \( c_n \) are given by

\[
c_n = \frac{1}{T} \int_{-T/2}^{T/2} \delta_p(t) e^{-jn\omega_0 t} \, dt = \frac{1}{T} \quad {\text{for all } n} \quad (2)
\]

(This is also a question on example sheet 6/6). We therefore have an alternative formula for the sampled signal:

\[
f_s(t) = f(t) \delta_p(t) = f(t) \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{jn\omega_0 t} \quad (3)
\]

\[
= \frac{1}{T} \sum_{n=-\infty}^{\infty} f(t) e^{jn\omega_0 t} \quad (4)
\]
It turns out that this formula is much easier to understand in the frequency domain. We will therefore determine the Fourier Transform of $f_s(t)$.

Looking at each term of the summation, we have from the frequency shift theorem:

$$f(t)e^{j\omega_0 t} \xleftrightarrow{FT} F(\omega - n\omega_0) \quad (5)$$

Hence the Fourier transform of the sum is (by linearity):

$$F_s(\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} F(\omega - n\omega_0) \quad (6)$$
i.e. The Fourier transform of the sampled signal is simply \( \frac{1}{T} \) times the Fourier transform of the continuous signal repeated every integer multiple of the sampling frequency and summed together.

**Example:**

![Spectrum repeated every integer multiple of the sampling frequency](image)

**Figure 6:** Spectrum repeated every integer multiple of the sampling frequency
We will see shortly that this fundamental result is all we need to answer the original question:

what is the optimal sampling frequency $1/T$ for perfect reconstruction of the original signal $f(t)$ from its samples $f(nT)$?
Discrete-time Fourier Transform

As well as the above analysis, note that the Fourier transform of the sampled signal can also be written as follows:

\[
F_s(\omega) = \int_{-\infty}^{\infty} f_s(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} \left\{ \sum_{n=-\infty}^{\infty} f(t) \delta(t - nT) \right\} e^{-j\omega t} dt
\]

\[
= \sum_{n=-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} f(t) e^{-j\omega t} \delta(t - nT) dt \right\}
\]

\[
= \sum_{n=-\infty}^{\infty} f(nT) e^{-jn\omega T}
\]

[Using sifting property of \( \delta \)]

This alternative formula is known as the **Discrete-Time Fourier Transform** or **DTFT**. The **DTFT** shows how to calculate the frequency content of the ideal sampled signal directly from its sampled values \( f(nT) \).
Nyquist Frequency and Reconstruction

We have seen that the spectrum (Fourier Transform) of a sampled signal consists of many repetitions of the spectrum of the original signal $f(t)$:

$$F_s(\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} F(\omega - n\omega_0)$$ (7)

Now, see Figure 7, which shows the sampled spectrum for a signal with bandwidth $\omega_{\text{max}}$ and for 3 possible values of the sampling frequency $\omega_0 = 2\pi / T$. 
Figure 7: The sampled spectrum $F_s(\omega)$ for various values of sampling frequency

(a) $\omega_0 = 4\omega_{\text{max}}$

(b) $\omega_0 = 2\omega_{\text{max}}$

(c) $\omega_0 = 1.5\omega_{\text{max}}$
Now, attempt to extract just $F(\omega)$ from the sampled spectrum $F_s(\omega)$

- Apply a filter with frequency response $H_r(\omega)$ (shown dotted) to the sampled signal.

- Since $Y(\omega) = H_r(\omega)F_s(\omega)$ we can see from the diagram that for cases (a) and (b), $Y(\omega) = F(\omega)$ and we have reconstructed the original signal spectrum perfectly. In the third case (c), $F(\omega)$ is not properly reconstructed.

- In general it is possible to reconstruct the original spectrum only when there is no overlap between the periodic repetitions of $F(\omega)$. 
The Nyquist sampling theorem can now be stated as:

If a signal $f(t)$ has a maximum frequency content (or bandwidth) $\omega_{\text{max}}$, then it is possible to reconstruct $f(t)$ perfectly from its sampled version $f_s(t)$ provided the sampling frequency is at least

$$\omega_0 = 2\omega_{\text{max}}, \text{ the ‘Nyquist frequency’}$$
• The minimum sampling frequency of $2 \times \omega_{max}$ is known as the Nyquist Frequency, $\omega_{Nyq}$.

• The repetitions of $F(\omega)$ in the sampled spectrum are known as aliasing.

• When a signal is sampled at a rate less than $\omega_{Nyq}$ the distortion due to the overlapping spectra is called aliasing distortion.
The ideal filter frequency response for perfect reconstruction is the rectangle pulse function:

\[
H_r(\omega) = \begin{cases} 
T, & -\omega_{max} < \omega < +\omega_{max} \\
0, & \text{otherwise}
\end{cases}
\]

The impulse response of the filter is then the inverse Fourier transform of \( H_r(\omega) \). We know that the inverse Fourier transform of a rectangular pulse is a \( \text{sinc} \) function, i.e.

\[
h_r(t) = \frac{\omega_{max} T}{\pi} \text{sinc}(\omega_{max} t)
\]
Or, if we are sampling exactly at the Nyquist frequency, \( \omega_{\text{max}} = \frac{\omega_0}{2} \), and the above becomes

\[
h_r(t) = \frac{\omega_0 T}{2\pi} \text{sinc}(\omega_0 t / 2) = \text{sinc}(\omega_0 t / 2)
\]

Since multiplication in the frequency domain implies convolution in the time domain, the equivalent recovery operation back in the time domain becomes:

\[
f(t) = \text{sinc}(\omega_0 t / 2) * f_s(t)
\]

(8)

By substituting \( f_s(t) = \sum_{n=-\infty}^{\infty} f(nT) \delta(t - nT) \) and performing the convolution, equation (8) becomes

\[
f(t) = \int_{-\infty}^{\infty} f_s(\tau) \text{sinc}(\omega_0(t - \tau)/2) d\tau
\]

\[
= \int_{-\infty}^{\infty} \left\{ \sum_{n=-\infty}^{\infty} f(nT) \delta(\tau - nT) \right\} \text{sinc}(\omega_0(t - \tau)/2) d\tau
\]
Swapping order of integral and summation gives

\[ f(t) = \sum_{n=-\infty}^{\infty} f(nT) \left\{ \int_{-\infty}^{\infty} \delta(\tau - nT) \text{sinc}(\omega_0(t - \tau)/2) d\tau \right\} \quad (9) \]

which can be evaluated using the sifting property as

\[ f(t) = \sum_{n=-\infty}^{\infty} f(nT) \text{sinc} \left[ \frac{\pi}{T} (t - nT) \right] \quad (10) \]

This can be viewed as an *exact* interpolation formula for determining \( f(t) \) from its samples \( f(nT) \).
Practical considerations

This is all very idealised. How would this be modified in practice?

- Must determine the maximum frequency component $\omega_{max}$ present in a signal before sampling.

- In order to eliminate the aliasing effects of high frequency noise or unwanted high signal frequencies, first filter the data with a low-pass filter having frequency response $H_r(\omega) / T$, i.e. just a unity gain lowpass filter - call this filtered signal $f(t)$ and proceed with sampling.

- Then perform sampling at the Nyquist rate $\omega_0 = \omega_{Nyq} = 2 \times \omega_{max}$ to give digital samples $f(nT)$.

- Reconstruct the signal by passing the sampled signal through the same filter $H_r(\omega)$. 
• In practice though we cannot exactly implement the ideal filter $H_r(\omega)$. Must therefore allow say 10% extra signal bandwidth for the transition band of the filters (see Paper 6 Communications course)
Example: A music signal has bandwidth 20kHz.

a) Determine the sampling period for this signal, assuming ideal filter responses.

b) Determine a suitable sampling rate assuming a realistically achievable filter response.

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