Complex-valued Wavelets, the Dual Tree, and Hilbert Pairs: why these lead to shift invariance and directional m-D wavelets?

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Important continuous-time complex-valued wavelets

The **Gabor** function

\[
\psi(t) = k \, e^{-t^2/2\sigma^2} \, e^{j\omega_0 t} \quad \hat{\psi}(\omega) = K \, e^{-\frac{1}{2}\sigma^2(\omega-\omega_0)^2}
\]

The **Morlet** wavelet

\[
\psi(t) = k \, e^{-t^2/2\sigma^2} \left( e^{j\omega_0 t} - \kappa_0 \right) \quad \hat{\psi}(\omega) = K \left( e^{-\frac{1}{2}\sigma^2(\omega-\omega_0)^2} - \kappa_0 \, e^{-\frac{1}{2}\sigma^2\omega^2} \right)
\]

The **Cauchy** wavelet

\[
\psi(t) = k \, (1 - j\beta t)^{-\alpha} \quad \hat{\psi}(\omega) = \begin{cases} 
K \omega^{\alpha-1} e^{-\omega/\beta}, & \omega \geq 0 \\
0, & \omega < 0 
\end{cases}
\]

Max gain is at \( \omega_0 = \beta(\alpha - 1) \).
Typically \( \alpha \approx 8 \) for one octave half-power bandwidth.
Plots of \( \psi(t) \) and \(|\hat{\psi}(\omega)|\) for Morlet, Cauchy (continuous) and dual-tree (discrete) complex wavelets

Note: \( k = (1 + j)/\sqrt{2} \); and the dual-tree filters are 18-tap Q-shift filters.
How can we produce *good* discrete wavelet transforms?

- What are the problems with real-valued wavelet bases?
- Why do we need the Dual Tree?
- What is the Hilbert Pair delay condition?
- Why does this give shift invariance?
- Why do we use Q-shift filters?
- How do we extend the dual-tree to multi-dimensions?
- Why do we get good directional filters in m-D?
- What are some applications of the DT CWT?
Real Discrete Wavelet Transform (DWT) in 1-D

Figure: (a) Tree of real filters for the DWT. (b) Reconstruction filter block for 2 bands at a time, used in the inverse transform.
Features of the (Real) Discrete Wavelet Transform (DWT)

- **Good compression** of signal energy.
- **Perfect reconstruction** with short support filters.
- **No redundancy** – hence orthonormal or bi-orthogonal transforms are possible.
- **Very low computation** – order-$N$ only.

But

- **Severe shift dependence**.
- **Poor directional selectivity** in 2-D, 3-D etc.

The DWT is normally implemented with a tree of highpass and lowpass filters, separated by 2 : 1 decimators.
Visualising Shift Invariance / Dependence

- Apply a standard input (e.g. unit step) to the transform for a range of shift positions.
- Select the transform coefficients from just one wavelet level at a time.
- Inverse transform each set of selected coefficients.
- Plot the component of the reconstructed output for each shift position at each wavelet level.
- Check for shift invariance (similarity of waveforms).

See Matlab demonstration / next slide.
Shift Invariance of DT CWT / Dependence of DWT

(a) Dual Tree CWT

(b) Real DWT

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Dual-Tree Wavelets and Hilbert pairs
UDRC 2013 (3)
Q-shift Dual Tree Complex Wavelet Transform (DT $\mathbb{C}$WT) in 1-D

Figure: Dual tree of real filters for the Q-shift $\mathbb{C}$WT, giving real and imaginary parts of complex coefficients from tree $a$ and tree $b$ respectively.
Features of the Dual Tree Complex Wavelet Transform

- **Good shift invariance** = **negligible aliasing**. Hence the transfer function through each subband is independent of shift and wavelet coefs can be interpolated within each subband, independent of all other subbands.

- **Good directional selectivity** in 2-D, 3-D etc. – derives from **analyticity** in 1-D (ability to separate positive from negative frequencies).

- **Perfect reconstruction** with short support filters.

- **Limited redundancy** – 2:1 in 1-D, 4:1 in 2-D etc.

- **Low computation** – much less than the undecimated (à trous) DWT.

Each tree contains purely real filters, but the two trees produce the **real and imaginary parts** respectively of each complex wavelet coefficient.
Dual-Tree CWT

Q-shift DT CWT Basis Functions – Levels 1 to 3

Basis functions for adjacent sampling points are shown dotted.
Why do we need the Dual Tree?

Making the wavelet responses **analytic** is a good way to **halve their bandwidth** and hence minimise aliasing.

**BUT** we cannot use complex filters in the DWT to obtain analyticity and perfect reconstruction together, because requirements conflict in the 2-band reconstruction block — analytic filters must suppress negative frequencies, while PR requires a flat overall frequency response.

So we use the **Dual Tree**:

- to create the **real** and **imaginary** parts of the analytic wavelets separately, using 2 trees of **purely real** filters;
- to efficiently synthesise a multiscale **shift-invariant** filterbank, with perfect reconstruction and **only 2:1 redundancy**;
- to produce complex coefficients whose **amplitude varies slowly** and whose **phase shift** depends approximately **linearly** on displacement.
What is the Hilbert Pair Delay Condition?

- **Given two parallel orthonormal discrete wavelet transforms, what is the constraint on the lowpass filters in each transform, such that the resulting continuous wavelets from each transform form a Hilbert Pair?**

  (This question and its answer are due to Ivan Selesnick in IEEE Signal Proc. Letters, June 2001.)

- A pair of wavelets, \( \psi_g(t) \) and \( \psi_h(t) \), are a **Hilbert Pair** if the complex function \( \psi_g(t) + j \psi_h(t) \) is **analytic** (i.e. its Fourier transform is zero for \( \omega < 0 \)).

- We shall show that this requires the lowpass filters, \( g_0(n) \) and \( h_0(n) \), of the two transforms to be related by the half-sample delay condition, expressed in the frequency domain as

\[
H_0(\omega) = e^{-j\omega/2}G_0(\omega)
\]
2-scale condition on Tree $a$ filters of a dyadic wavelet transform

**Scaling function:**
\[ \phi_g(t) = 2 \sum_n g_0(n) \phi_g(2t - n) \quad (1) \]

**Mother wavelet:**
\[ \psi_g(t) = 2 \sum_n g_1(n) \phi_g(2t - n) \quad (2) \]

Taking the Fourier transform of (1) gives the frequency domain relationship

\[ \hat{\phi}_g(\omega) = \int_{-\infty}^{\infty} 2 \sum_n g_0(n) \phi_g(2t - n) e^{-j\omega t} \, dt \]

\[ = \int_{-\infty}^{\infty} \sum_n g_0(n) \phi_g(u) e^{-j\omega u/2} e^{-j\omega n/2} \, du \quad \text{where} \ u = 2t - n \]

\[ = \sum_n g_0(n) e^{-j\omega n/2} \cdot \int_{-\infty}^{\infty} \phi_g(u) e^{-j\omega u/2} \, du \]

\[ = G_0(\frac{\omega}{2}) \cdot \hat{\phi}_g(\frac{\omega}{2}) \quad (3) \]
Infinite Product formulae

Iterating on (3):
\[ \hat{\phi}_g(\omega) = G_0\left(\frac{\omega}{2}\right) G_0\left(\frac{\omega}{4}\right) \hat{\phi}_g\left(\frac{\omega}{4}\right) = \cdots = \prod_{k=1}^{\infty} G_0(2^{-k}\omega) \hat{\phi}_g(0) \] (4)

Similarly, from (2) and (4):
\[ \hat{\psi}_g(\omega) = G_1\left(\frac{\omega}{2}\right) \hat{\phi}_g\left(\frac{\omega}{2}\right) = G_1\left(\frac{\omega}{2}\right) \prod_{k=2}^{\infty} G_0(2^{-k}\omega) \hat{\phi}_g(0) \] (5)

And similarly, for the Tree \( b \) filters:
\[ \hat{\phi}_h(\omega) = \prod_{k=1}^{\infty} H_0(2^{-k}\omega) \hat{\phi}_h(0) \] (6)
\[ \hat{\psi}_h(\omega) = H_1\left(\frac{\omega}{2}\right) \prod_{k=2}^{\infty} H_0(2^{-k}\omega) \hat{\phi}_h(0) \] (7)

The amplitude scaling of \( \phi_g(t) \) and \( \phi_h(t) \) is arbitrary, so we choose \( \hat{\phi}_g(0) = \hat{\phi}_h(0) = 1 \) to give them both unit area.
Conjugate Quadrature Filterbank (CQF)

In an **orthonormal** wavelet transform, \( G_1 \) and \( G_0 \) **form a CQF** (also known as a Quadrature Mirror Filterbank, QMF), such that

\[
G_1(\omega) = e^{-jm\omega} G_0^*(\omega \pm \pi)
\]

where we use \( \pm \pi \) to emphasise the \( 2\pi \)-periodic nature of \( G_0 \) and \( G_1 \), and \( G_0^* \) means complex conjugate of \( G_0 \). The delay shift of \( m \) samples must be an odd integer and is usually chosen to approximately equalise the group delay or the support of \( G_0 \) and \( G_1 \).

Similarly

\[
H_1(\omega) = e^{-jm\omega} H_0^*(\omega \pm \pi)
\]

Hence we can now express the wavelet frequency responses, \( \hat{\psi}_g(\omega) \) and \( \hat{\psi}_h(\omega) \), purely in terms of the two lowpass filters \( G_0 \) and \( H_0 \).
The Hilbert Pair Condition

This condition is

\[
\frac{\hat{\psi}_h(\omega)}{\hat{\psi}_g(\omega)} = \begin{cases} 
  j & \text{if } \omega < 0 \\
  -j & \text{if } \omega > 0
\end{cases}
\] (10)

Note that the behaviour of the RHS near \( \omega = 0 \) is immaterial, because, for wavelets to be admissible bandpass functions, \( \hat{\psi}_g(0) = \hat{\psi}_h(0) = 0 \).

Substituting (8) into (5) and (9) into (7), we get the following expression for this ratio

\[
\frac{\hat{\psi}_h(\omega)}{\hat{\psi}_g(\omega)} = e^{-jm\omega/2} \frac{H_0^*(\omega/2 \pm \pi)}{e^{-jm\omega/2} G_0^*(\omega/2 \pm \pi)} \left[ \prod_{k=2}^\infty H_0(2^{-k}\omega) \right] \hat{\phi}_h(0) \\
\frac{\hat{\phi}_h(0)}{\hat{\phi}_g(0)} = R_0^*(\omega/2 \pm \pi) \left[ \prod_{k=2}^\infty R_0(2^{-k}\omega) \right]
\] (11)

where \( R_0(\omega) = H_0(\omega)/G_0(\omega) \) and is \( 2\pi \)-periodic. \( R_0 \) will give the desired relationship between \( H_0 \) and \( G_0 \) if (10) and (11) are both satisfied.
Hilbert-pair delay condition

Solving for $R_0(\omega)$

Hence

$$R_0^*(\omega/2 \pm \pi) \left[ \prod_{k=2}^{\infty} R_0(2^{-k}\omega) \right] = \begin{cases} j & \text{if } \omega < 0 \\ -j & \text{if } \omega > 0 \end{cases} \quad (12)$$

Since the modulus of the RHS of (12) is unity everywhere, and the LHS contains an infinite product of terms in $R_0$, which will tend to grow or shrink if $R_0$ does not have unit magnitude, we deduce that $|R_0(\omega)| = 1$.

Now consider the phase $\theta(\omega)$ of $R_0$, by letting

$$R_0(\omega) = e^{j\theta(\omega)} \quad (13)$$

Equating the phases in (12), we require that

$$-\theta(\omega/2 \pm \pi) + \sum_{k=2}^{\infty} \theta(2^{-k}\omega) = \begin{cases} \pi/2 & \text{if } \omega < 0 \\ -\pi/2 & \text{if } \omega > 0 \end{cases} \quad (14)$$
Deducing the form of $\theta(\omega)$

Because of the infinite sum in (14), we require $\theta(\omega) \to 0$ as $\omega \to 0$. Hence $\theta(0) = 0$.

Since $g_0(n)$ and $h_0(n)$ are purely real and lowpass, $R_0(\omega)$ must be conjugate symmetric and so $\theta(\omega)$ must be a continuous odd function about $\omega = 0$.

It can be shown (by Fourier analysis on $\theta'(\omega)$) that any non-linear terms in $\theta(\omega)$ would prevent (14) from being satisfied, because in (14) the gradient of the first term must cancel out the gradient of the summation terms at all $\omega \neq 0$.

Therefore we let

$$\theta(\omega) = \alpha \omega \quad \text{for } -\pi < \omega < \pi, \text{ where } \alpha \text{ is a constant.}$$

Hence

$$\theta\left(\frac{\omega}{2} \pm \pi\right) = \begin{cases} \alpha\left(\frac{\omega}{2} + \pi\right) & \text{if } -4\pi < \omega < 0 \\ \alpha\left(\frac{\omega}{2} - \pi\right) & \text{if } 0 < \omega < 4\pi \end{cases}$$

Note that $\theta(\omega)$ must be $2\pi$-periodic for $|\omega| \geq \pi$, and so it will have discontinuities at $\omega = \pm\pi$, if $\alpha$ is not an integer.

These become discontinuities at $\omega = 0$ in $\theta\left(\frac{\omega}{2} \pm \pi\right)$. 
Typical plots of $\theta(\omega)$ and terms in equ.(14)

$$\theta(\omega) \approx \alpha \omega, \quad -\pi < \omega < \pi$$

and $\theta(\omega)$ is $2\pi$-periodic

$$-\theta\left(\frac{\omega}{2} \pm \pi\right)$$

$$\theta\left(\frac{\omega}{4}\right)$$

$$\theta\left(\frac{\omega}{8}\right)$$

$$-\theta\left(\frac{\omega}{2} \pm \pi\right) + \sum_{k=2}^{\infty} \theta(2^{-k}\omega)$$
Calculating $\alpha$

Noting that $\sum_{k=2}^{\infty} \theta(2^{-k}\omega) = \alpha \omega \left[\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots\right] = \frac{\alpha \omega}{2}$ if $-4\pi < \omega < 4\pi$,

and substituting (16) into (14) gives

$$-\alpha(\frac{\omega}{2} + \pi) + \frac{\alpha \omega}{2} = \frac{\pi}{2}$$ \text{ if } -4\pi < \omega < 0 \hspace{1cm} (17)

and

$$-\alpha(\frac{\omega}{2} - \pi) + \frac{\alpha \omega}{2} = -\frac{\pi}{2}$$ \text{ if } 0 < \omega < 4\pi \hspace{1cm} (18)

(17) and (18) are both satisfied if $\alpha = -\frac{1}{2}$, and therefore

$$\frac{H_0(\omega)}{G_0(\omega)} = R_0(\omega) = e^{j\theta(\omega)} = e^{j\alpha \omega} = e^{-j\omega/2} \text{ for } -\pi < \omega < \pi \hspace{1cm} (19)$$

This is the half-sample delay solution that makes $\psi_h(t)$ the Hilbert transform of $\psi_g(t)$.

Ozkaramanli and Yu (Dec 2005 and June 2006) have shown this solution to be unique and applicable to biorthogonal as well as orthonormal wavelet transforms.
Fig. 1: Dual tree of real filters for the Q-shift CWT, giving real and imaginary parts of complex coefficients from tree $a$ and tree $b$ respectively. Figures in brackets indicate the approximate delay for each filter, where $q = \frac{1}{4}$ sample period. Special level 1 filters, $G^1$ and $H^1$, allow for the finite number of levels.
Why does the delay condition give shift invariance?

- The **half-sample delay** between the $G_0$ and $H_0$ lowpass filters means that their output samples are uniformly interleaved at all scales, and hence the sample rate is effectively doubled everywhere.

- The doubled sampling rate is sufficient to virtually **eliminate aliasing** if filters of 12 or more taps are used.

- If aliasing is eliminated in the **lowpass** branch of each 2-band reconstruction block, then it must also be eliminated in the **highpass** branch (since perfect reconstruction is achieved).

- Elimination of aliasing means that each subband can be represented by a **unique z-transfer function**, and hence the filtering is **LTI**, linear time-invariant (i.e. shift-invariant).

At level 1 of a finite dual tree, the delay difference must **increase to one sample** to compensate for the absence of delay differences at finer levels.
Why do we use Q-shift filters (below level 1)?

- Half-sample delay difference is obtained with filter delays of \( \frac{1}{4} \) and \( \frac{3}{4} \) of a sample period (instead of 0 and \( \frac{1}{2} \) a sample for our original DT CWT).
- This is achieved with an asymmetric even-length filter \( G_0(z) \) and its time reverse \( H_0(z) = z^{-1} G_0(z^{-1}) \). \( G_1(z) \) and \( H_1(z) \) are the CQFs of these.
- Due to the asymmetry (like Daubechies filters), these may be designed to give an orthonormal perfect reconstruction wavelet transform in each tree.
- Tree \( b \) filters are the reverse of tree \( a \) filters, and reconstruction filters are the reverse of analysis filters, so all filters are from the same orthonormal set, yielding a tight-frame transform.
- Both trees have the same frequency responses (in magnitude).
- The combined complex impulse responses are conjugate symmetric about their mid points, even though the separate responses are asymmetric. Hence symmetric extension still works at image edges.

At level 1, any DWT filters can be used.
Basis functions for adjacent sampling points are shown dotted.
Frequency Responses of 18-tap Q-shift filters

Wavelets at level:

Scaling fn. at level 4
The DT CWT in 2-D – why good directional selectivity?

When the DT CWT is applied to 2-D signals (images), it has the following features:

- It is performed separably, using 2 trees for the rows of the image and 2 trees for the columns – yielding a **Quad-Tree** structure (4:1 redundancy).

- The 4 quad-tree components of each coefficient are combined by simple sum and difference operations to yield a **pair of complex coefficients**. These are part of two separate subbands in adjacent quadrants of the 2-D spectrum.

- This produces **6 directionally selective subbands** at each level of the 2-D DT CWT. The next slide shows basis functions of these subbands at level 4, and compares them with the 3 subbands of a 2-D DWT.

- The DT CWT is directionally selective because the complex filters can **separate positive and negative frequency components** in 1-D, and hence **separate adjacent quadrants** of the 2-D spectrum. Real separable filters cannot do this!
Basis functions of 2-D Q-shift complex wavelets (top), and of 2-D real wavelet filters (bottom), all illustrated at level 4 of the transforms. The complex wavelets provide 6 directionally selective filters, while real wavelets provide 3 filters, only two of which have a dominant direction. The 1-D bases, from which the 2-D complex bases are derived, are shown to the right.
Frequency Responses of 2-D Q-shift filters at levels 3 and 4

Contours shown at −1 dB and −3 dB.
Q-shift Filter Design Requirements

![Filter Diagram](image)

**Fig 2: 2-band analysis and reconstruction filter bank.**

- **No aliasing:** \( G_1(z) = zH_0(-z) \); \( H_1(z) = z^{-1}G_0(-z) \)
- **Perfect reconstruction:** \( H_0(z)G_0(z) + H_0(-z)G_0(-z) = 2 \)
- **Orthogonality:** \( G_0(z) = z^{-k}H_0(z^{-1}) \)
- **Group delay** \( \simeq \frac{1}{4} \) **sample period** for \( H_0 \).
- **Good smoothness** properties when iterated over scale.
To get $2n$-tap lowpass filters, $H_0(z)$ and $G_0(z)$, with $\frac{1}{4}$ and $\frac{3}{4}$ sample delays:

- Design a $4n$-tap symmetric lowpass filter $H_{L2}(z)$ with half the required bandwidth and a delay of $\frac{1}{2}$ sample;
- **Subsample** $H_{L2}(z)$ by 2:1 to get $H_0(z)$ and $G_0(z)$.

Fig 3: Impulse response of $H_{L2}(z)$ for $n = 6$. The $H_0$ and $G_0$ filter taps are shown as circles and crosses respectively.
Filter Design – Perfect Reconstruction (PR)

For PR and orthogonality, $H_0(z) G_0(z) = H_0(z) H_0(z^{-1})$ must have no terms in $z^{2k}$ except the term in $z^0$.

Therefore $H_0(z^2) H_0(z^{-2})$ must have no terms in $z^{4k}$ except the term in $z^0$.

But

$$H_{L2}(z) = H_0(z^2) + z^{-1} H_0(z^{-2})$$

and so

$$H_{L2}(z) H_{L2}(z^{-1}) = 2 H_0(z^2) H_0(z^{-2}) + \underbrace{z^{-1} H_0^2(z^{-2}) + z H_0^2(z^2)}_{\text{odd powers of } z \text{ only}}$$

Therefore $H_{L2}(z) H_{L2}(z^{-1})$ must have no terms in $z^{4k}$ except the term in $z^0$.

Hence we can include PR as a direct design constraint on $H_{L2}(z) H_{L2}(z^{-1})$. 
To obtain smoothness when iterated over many scales:

- Ensure that the stopband of $H_0(z)$ suppresses energy at frequencies where unwanted passbands appear from subsampled filters operating at coarser scales.

Consider the combined frequency response of $H_0$ over just two scales:

$$H_0(z) H_0(z^2) |_{z=e^{j\omega}} = H_0(e^{j\omega}) H_0(e^{2j\omega})$$

If the stopband of $H_0(e^{j\omega})$ covers $\omega_s \leq \omega \leq \pi$, then the unwanted transition band and passband of $H_0(e^{2j\omega})$ will extend from $\pi - \frac{\omega_s}{2}$ to $\pi$.

For $H_0(e^{j\omega})$ to suppress the unwanted bands of $H_0(e^{2j\omega})$:

$$\omega_s \leq \pi - \frac{\omega_s}{2} \quad \text{and so} \quad \omega_s \leq \frac{2\pi}{3} \quad \text{(see fig 4)}$$
Fig 4: Frequency responses of $H_L^2(z)$ (blue), $H_0(z)$ (green), $H_0(z) H_0(z^2)$ (magenta), and the gain correction matrix $T$ (red) for $n = 6$ (12 taps for $H_0$).
Optimization for MSE in the frequency domain

We have now reduced the ideal design conditions for the length $4n$ symmetric lowpass filter $H_{L2}$ to be:

- Zero amplitude for all the terms of $H_{L2}(z) H_{L2}(z^{-1})$ in $z^{4k}$ except the term in $z^0$, which must be 1 (these are quadratic constraints on coef vector $h_{L2}$);
- Zero (or near-zero) amplitude of $H_{L2}(e^{j\omega})$ for the stopband, $\frac{\pi}{3} \leq \omega \leq \pi$ (these are linear constraints on $h_{L2}$).

If all constraints were linear, the LMS error solution for $h_{L2}$ could be found using a matrix pseudo-inverse method. Therefore we linearise the problem and iterate.

If $h_{L2}$ at iteration $i$ is $h_i = h_{i-1} + \Delta h_i$, then

$$h_i \ast h_i = (h_{i-1} + \Delta h_i) \ast (h_{i-1} + \Delta h_i) = h_{i-1} \ast (h_{i-1} + 2\Delta h_i) + \Delta h_i \ast \Delta h_i$$

Since $\Delta h_i$ becomes small as $i$ increases, the final term can be neglected and the convolution ($\ast$) is expressed as a linear function of $\Delta h_i$. 
Hence we solve for $\Delta h_i$ such that:

$$\begin{align*}
C_{i-1} (h_{i-1} + 2\Delta h_i) &= [0 \ldots 0 1]^T \\
F (h_{i-1} + \Delta h_i) &\approx [0 \ldots 0]^T
\end{align*}$$

where $C_{i-1}$ calculates every 4th term in the convolution with $h_{i-1}$, and $F$ evaluates the Fourier transform at $M$ discrete frequencies $\omega$ from $\frac{\pi}{3}$ to $\pi$ (typically $M \approx 8n$).

Note that only one side of the symmetric convolution is needed in the rows of $C_{i-1}$, and the columns of $C_{i-1}$ and $F$ can be combined in pairs so that only the first half of the symmetric $\Delta h_i$ need be solved for.

To obtain **high accuracy solutions to the PR constraints**, we scale the equations in $C_{i-1}$ up by $\beta_i = 2^i$ to get the following iterative LMS method for $\Delta h_i$ and then $h_i$:

$$\begin{pmatrix}
2\beta_i C_{i-1} \\
F
\end{pmatrix} \Delta h_i = \begin{pmatrix}
\beta_i (c - C_{i-1} h_{i-1}) \\
-F h_{i-1}
\end{pmatrix}$$

and $h_i = h_{i-1} + \Delta h_i$

where $c = [0 \ldots 0 1]^T$. 
Q-shift filter design

Two final refinements

- To include **transition band** effects, we scale rows of $F$ by diagonal matrix $T_i$, the gain (at iteration $i$) of $H_0(z^2)/H_0(1)$ at frequencies corresponding to $\frac{\pi}{3} \leq \omega \leq \frac{\pi}{2}$ in the frequency domain of $H_{L2}$ ($T_i$ is the red curve in fig 4).

- To insert **predefined zeros** in $H_0(z)$ or $H_{L2}(z)$, we first note that a zero at $z = e^{j\pi}$ in $H_0$ will be produced by a pair of zeros at $z = e^{\pm j\pi/2}$ in $H_{L2}$. We can force zeros in $H_{L2}$ by forming a convolution matrix $H_f$ such that $H_f \ h'_i = h_i$, where $h'_i$ is the coef vector of the filter which represents all the zeros of $H_{L2}$ that are not predefined, and $H_f$ produces convolution with the predefined zeros.

Hence we now solve for $\Delta h'_i$ and then $h_i$ using

$$
\begin{bmatrix}
2\beta_i C_{i-1} \\
T_{i-1} F
\end{bmatrix}
H_f \ \Delta h'_i =
\begin{bmatrix}
\beta_i (c - C_{i-1} h_{i-1}) \\
-T_{i-1} F \ h_{i-1}
\end{bmatrix}
with \ h_i = h_{i-1} + H_f \ \Delta h'_i
$$
Fig 5: Frequency responses of $H_{L2}(z)$ for $n = 8$ (blue), $n = 12$ (green) and $n = 16$ (red). Each filter has one predefined zero at $\omega = \frac{\pi}{2}$ and one at $\omega = \pi$. 
Initiation

To initialise the iterative algorithm when $i = 1$, we must define $h_0$ and hence $C_0$ and $T_0$. This is not critical and can be achieved by a simple inverse FFT of an ‘ideal’ lowpass frequency response for $H_{L2}(e^{j\omega})$ with a root-raised-cosine transition band covering the range

$$\frac{\pi}{6} < \omega < \frac{\pi}{3}$$

The impulse response is truncated symmetrically to length $4n$ to obtain $h_0$. $C_0$ and $T_0$ may then be calculated from $h_0$.

Convergence

For some larger values of $n$, convergence can be slow. We have found this can be improved by using

$$h_i = h_{i-1} + \alpha H_f \Delta h'_i$$

where $0 < \alpha < 1$ (e.g. $\alpha \sim 0.8$)
Filter Design Results

- Figs. 4 and 5 show the frequency responses of $H_{L2}(z)$ for the cases $n = 6, 8, 12$ and $16$, when there is one predefined zero at $\omega = \frac{\pi}{2}$ and one at $\omega = \pi$.

- Figs. 6 to 15 show, for a range of values of $n$, the impulse response of $H_{L2}(z)$, the level-4 DT CWT scaling functions and wavelets, the frequency responses of $H_0(z)$ and of $H_0(z)H_0(z^2)$, and the group delay of $H_0(z)$.

- Figs. 6 to 11 show these responses for the cases $n = 5, 6$ and $7$, with either 0 or 1 predefined zero in $H_0(z)$ at $\omega = \pi$.

- Figs. 12 to 15 show these responses for the cases $n = 8, 12$ and $16$, with 1 predefined zero in $H_0(z)$ at $\omega = \pi$.

Note how the responses improve with increasing $n$. The effect of predefining a zero in $H_0$ is in general quite small. More predefined zeros tend to degrade performance. $n = 7$ gives a good tradeoff between complexity and performance.
Fig 6: Q-shift filters for $n = 5$ (10 filter taps) and no predefined zeros.
Fig 7: Q-shift filters for \( n = 5 \) (10 filter taps) and 1 predefined zero at \( \omega = \pi \).
Fig 8: Q-shift filters for $n = 6$ (12 filter taps) and no predefined zeros.
Fig 9: Q-shift filters for $n = 6$ (12 filter taps) and 1 predefined zero at $\omega = \pi$. 

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Fig 10: Q-shift filters for $n = 7$ (14 filter taps) and no predefined zeros.
Fig 11: Q-shift filters for $n = 7$ (14 filter taps) and 1 predefined zero at $\omega = \pi$. 

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Fig 12: Q-shift filters for $n = 8$ (16 filter taps) and 1 predefined zero at $\omega = \pi$. 
Fig 13: Q-shift filters for $n = 10$ (20 filter taps) and 1 predefined zero at $\omega = \pi$. 

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Fig 14: Q-shift filters for $n = 12$ (24 filter taps) and 1 predefined zero at $\omega = \pi$. 

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Fig 15: Q-shift filters for $n = 16$ (32 filter taps) and 1 predefined zero at $\omega = \pi$. 
Filter Design – Conclusions

- The proposed algorithm gives a fast and effective way of designing Q-shift filters for the DT CWT.
- All filters produce perfect reconstruction, tight frames and linear-phase complex wavelets.
- As the length of the filters \((2n)\) increases, the design method gives improvements in stopband attenuation, constancy of group delay, and smoothness in the resulting wavelet bases. Hence we get increasing accuracy of shift-invariance.
- The algorithm works well for filter lengths from 10 to over 50 taps.
- Matlab code for the algorithm and papers on the DT CWT can be downloaded from the author’s website, http://www-sigproc.eng.cam.ac.uk/~ngk/.
- Matlab code to implement the DT CWT is free for researchers and available by emailing the author at ngk@eng.cam.ac.uk.
Shift Invariance

Visualising Shift Invariance / Dependence

- Apply a standard input (e.g., unit step) to the transform for a **range of shift positions**.
- Select the transform coefficients from **just one wavelet level** at a time.
- Inverse transform each set of selected coefficients.
- Plot the component of the reconstructed output for each shift position at each wavelet level.
- Check for **shift invariance** (similarity of waveforms).

See Matlab demonstration / next slide.
Shift Invariance of DT CWT / Dependence of DWT

(a) Dual Tree CWT

Input
Wavelets
Level 1
Level 2
Level 3
Level 4
Scaling fn
Level 4

DT CWT, $n = 9$ (18-tap filters)

(b) Real DWT

Input
Wavelets
Level 1
Level 2
Level 3
Level 4
Scaling fn
Level 4

Real DWT, (13,19-tap filters)
Shift Invariance of simpler DT CWTs

DT CWT, $n = 7$ (14-tap filters)

DT CWT, $n = 5$ (10-tap filters)
Shift Invariance – quantitative measurement

Letting $W = e^{j2\pi/M}$, multi-rate analysis gives:

$$Y(z) = \frac{1}{M} \sum_{k=0}^{M-1} X(W^k z)[A(W^k z) C(z) + B(W^k z) D(z)]$$

For shift invariance, aliasing terms ($k \neq 0$) must be negligible. So we design $B(W^k z) D(z)$ to cancel $A(W^k z) C(z)$ for all non-zero $k$ that give overlap of the passbands of filters $C(z)$ or $D(z)$ with those of shifted filters $A(W^k z)$ or $B(W^k z)$.
A Measure of Shift Invariance – Aliasing Energy Ratio $R_a$

Since

$$Y(z) = \frac{1}{M} \sum_{k=0}^{M-1} X(W^k z)[A(W^k z) C(z) + B(W^k z) D(z)]$$

we quantify the shift dependence of a transform by calculating the ratio of the total energy of the unwanted aliasing transfer functions (the terms with $k \neq 0$) to the energy of the wanted transfer function (when $k = 0$):

$$R_a = \frac{\sum_{k=1}^{M-1} \mathcal{E}\{A(W^k z) C(z) + B(W^k z) D(z)\}}{\mathcal{E}\{A(z) C(z) + B(z) D(z)\}}$$

where $\mathcal{E}\{U(z)\}$ calculates the energy, $\sum_r |u_r|^2$, of the impulse response of a $z$-transfer function, $U(z) = \sum_r u_r z^{-r}$.

$\mathcal{E}\{U(z)\}$ may also be interpreted in the frequency domain as the integral of the squared magnitude of the frequency response, $\frac{1}{2\pi} \int_{-\pi}^{\pi} |U(e^{j\theta})|^2 \, d\theta$ from Parseval’s theorem.
Types of DT CWT filters

We show results for the following combinations of filters:

A: \((13,19)\)-tap near-orthogonal filters at level 1, 18-tap Q-shift filters at levels \(\geq 2\). (Most complex)

B: \((13,19)\)-tap near-orthogonal filters at level 1, 14-tap Q-shift filters at levels \(\geq 2\).

C: \((9,7)\)-tap bi-orthogonal filters at level 1, 18-tap Q-shift filters at levels \(\geq 2\).

D: \((9,7)\)-tap bi-orthogonal filters at level 1, 14-tap Q-shift filters at levels \(\geq 2\).

E: \((9,7)\)-tap bi-orthogonal filters at level 1, 6-tap Q-shift filters at levels \(\geq 2\).

F: \((5,3)\)-tap bi-orthogonal filters at level 1, 6-tap Q-shift filters at levels \(\geq 2\). (Least complex)
### Aliasing Energy Ratios

Values of $R_a$ in dB, for filter types A to F over levels 1 to 5:

<table>
<thead>
<tr>
<th>Filters:</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>DWT</th>
</tr>
</thead>
<tbody>
<tr>
<td>Complexity:</td>
<td>13,19+18</td>
<td>13,19+14</td>
<td>9,7+18</td>
<td>9,7+14</td>
<td>9,7+6</td>
<td>5,3+6</td>
<td>13,19</td>
</tr>
<tr>
<td>Filters:</td>
<td>2.3</td>
<td>2.0</td>
<td>1.9</td>
<td>1.6</td>
<td>1.0</td>
<td>0.7</td>
<td>1.0</td>
</tr>
<tr>
<td><strong>Wavelet</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Level 1</td>
<td>$-\infty$</td>
<td>$-\infty$</td>
<td>$-\infty$</td>
<td>$-\infty$</td>
<td>$-\infty$</td>
<td>$-\infty$</td>
<td>$-9.40$</td>
</tr>
<tr>
<td>Level 2</td>
<td>$-31.40$</td>
<td>$-29.06$</td>
<td>$-22.96$</td>
<td>$-21.81$</td>
<td>$-18.49$</td>
<td>$-14.11$</td>
<td>$-3.54$</td>
</tr>
<tr>
<td>Level 3</td>
<td>$-27.93$</td>
<td>$-25.10$</td>
<td>$-20.32$</td>
<td>$-18.96$</td>
<td>$-14.60$</td>
<td>$-11.00$</td>
<td>$-3.53$</td>
</tr>
<tr>
<td><strong>Scaling fn.</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Level 1</td>
<td>$-\infty$</td>
<td>$-\infty$</td>
<td>$-\infty$</td>
<td>$-\infty$</td>
<td>$-\infty$</td>
<td>$-\infty$</td>
<td>$-9.40$</td>
</tr>
<tr>
<td>Level 3</td>
<td>$-35.88$</td>
<td>$-29.21$</td>
<td>$-36.94$</td>
<td>$-29.33$</td>
<td>$-21.75$</td>
<td>$-20.63$</td>
<td>$-9.37$</td>
</tr>
</tbody>
</table>

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The **Dual-Tree Complex Wavelet Transform** provides:

- **Approximate shift invariance**
- **Directionally selective** filtering in 2 or more dimensions
- **Low redundancy** – only $2^m : 1$ for $m$-D signals
- **Perfect reconstruction**
- **Orthonormal filters** below level 1, but still giving **linear phase** (conjugate symmetric) complex wavelets
- **Low computation** – order-$N$; less than $2^m$ times that of the fully decimated DWT ($\sim 3.3$ times in 2-D, $\sim 5.1$ times in 3-D)
- **A general purpose multi-resolution front-end** for many image analysis and reconstruction tasks . . . (see next slide)
A general purpose multi-resolution front-end for many image analysis and reconstruction tasks:
- Tight-frame sparse representation of images and 3D data
- Denoising
- Enhancement (deconvolution)
- Motion / displacement estimation and compensation
- Registration in 2D or 3D.
- Texture analysis / synthesis
- Segmentation and classification
- Feature-point detection and rotation-invariant description
- Object recognition and tracking

Papers on complex wavelets are available at:
[http://www.eng.cam.ac.uk/~ngk/](http://www.eng.cam.ac.uk/~ngk/)

A Matlab DTCWT toolbox is available on request from:
gkg@eng.cam.ac.uk