# DUAL TREE COMPLEX WAVELETS
# PART 1

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# Dual Tree Complex Wavelets

## Part 1:

- Basic form of the DT CWT
- How it achieves shift invariance
- DT CWT in 2-D and 3-D – directional selectivity
- Application to image denoising

## Part 2:

- Q-shift filter design
- How good is the shift invariant approximation
- Further applications – regularisation, registration, object recognition, watermarking.

# Features of the (Real) Discrete Wavelet Transform (DWT)

- **Good compression** of signal energy.
- **Perfect reconstruction** with short support filters.
- **No redundancy**.
- **Very low computation** – order-$N$ only.

But

- **Severe shift dependence**.
- **Poor directional selectivity** in 2-D, 3-D etc.

The DWT is normally implemented with a tree of highpass and lowpass filters, separated by $2:1$ decimators.

# Real Discrete Wavelet Transform (DWT) in 1-D

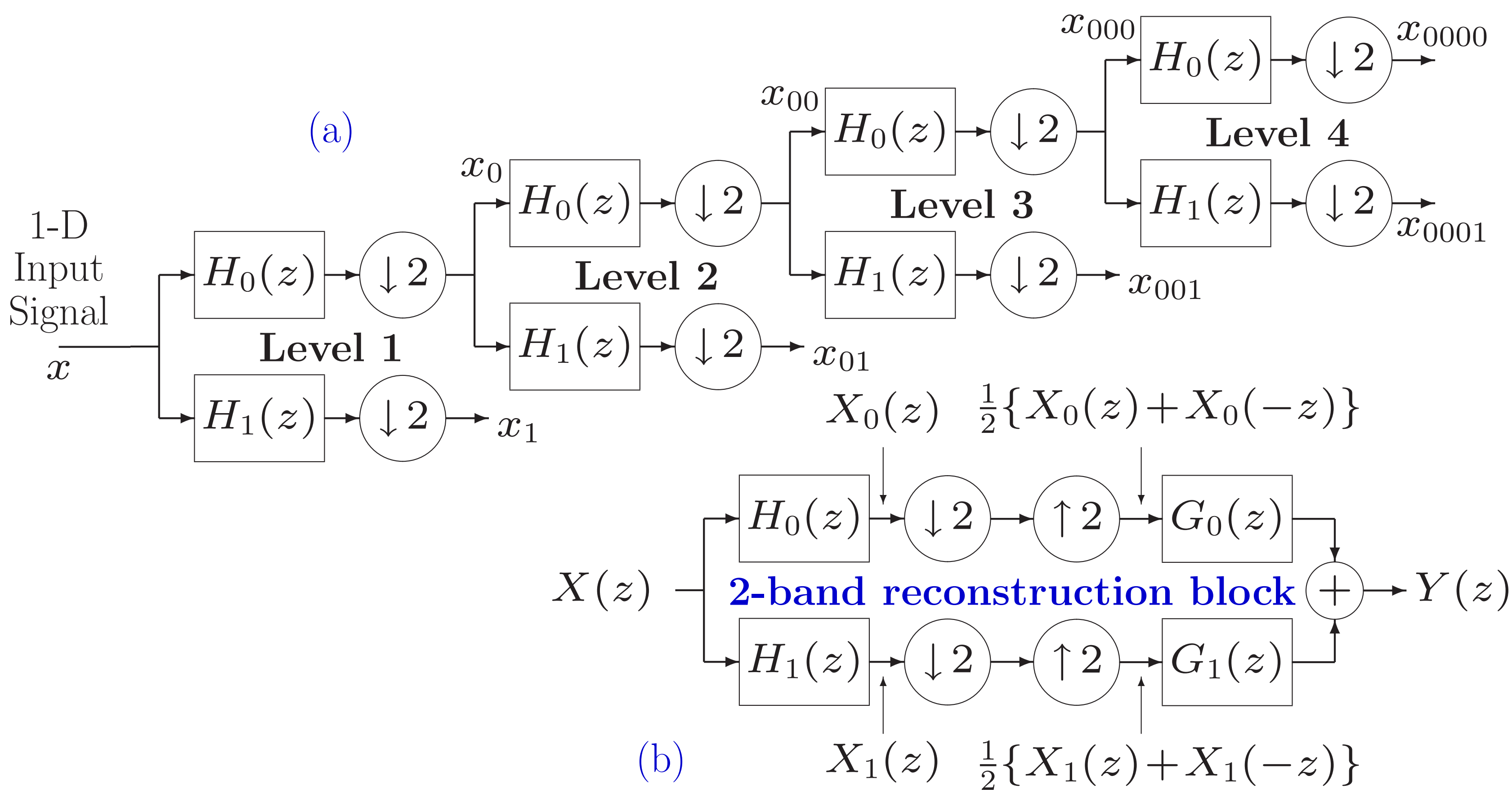



Figure 1: (a) Tree of real filters for the DWT. (b) Reconstruction filter block for 2 bands at a time, used in the inverse transform.

## Visualising Shift Invariance

- Apply a standard input (e.g. unit step) to the transform for a **range of shift positions**.
- Select the transform coefficients from **just one wavelet level** at a time.
- Inverse transform each set of selected coefficients.
- Plot the component of the reconstructed output for each shift position at each wavelet level.
- Check for **shift invariance** (similarity of waveforms).

See Matlab demonstration.

# FEATURES OF THE DUAL TREE COMPLEX WAVELET TRANSFORM (DT CWT)

- Good **shift invariance**.
- Good **directional selectivity** in 2-D, 3-D etc.
- **Perfect reconstruction** with short support filters.
- **Limited redundancy** – 2:1 in 1-D, 4:1 in 2-D etc.
- **Low computation** – much less than the undecimated (à trous) DWT.

Each tree contains purely real filters, but the two trees produce the **real and imaginary parts** respectively of each complex wavelet coefficient.

# Q-shift Dual Tree Complex Wavelet Transform in 1-D

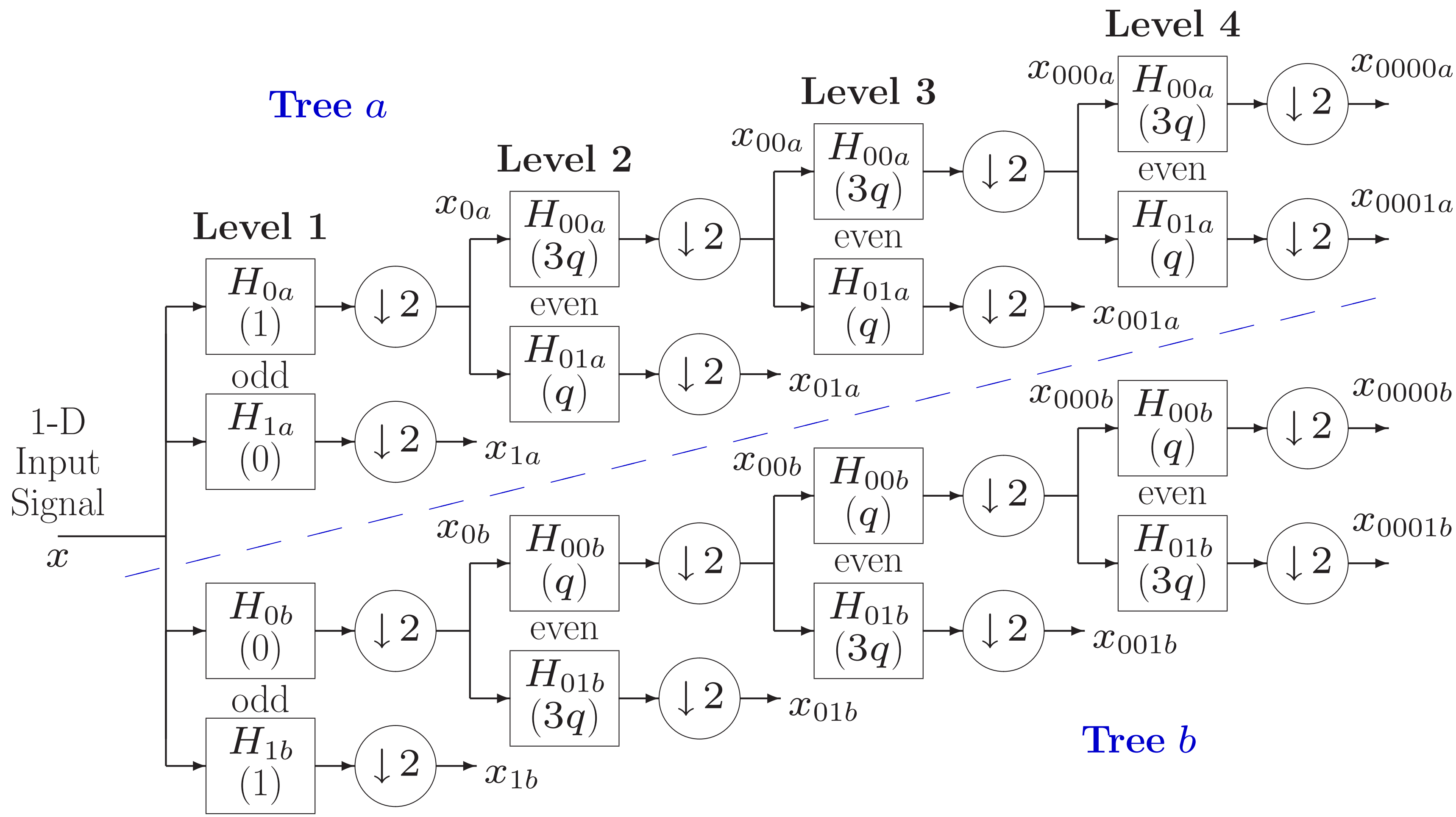



Figure 2: Dual tree of real filters for the Q-shift CWT, giving real and imaginary parts of complex coefficients from tree $a$ and tree $b$ respectively. Figures in brackets indicate the approximate delay for each filter, where $q = \frac{1}{4}$ sample period.

## Features of the Q-shift Filters

Below level 1:

- Half-sample delay difference is obtained with filter delays of $\frac{1}{4}$ and $\frac{3}{4}$ of a sample period (instead of 0 and $\frac{1}{2}$ a sample for our original DT CWT).
- This is achieved with an **asymmetric even-length** filter $H(z)$ and its time reverse $H(z^{-1})$.
- Due to the asymmetry (like Daubechies filters), these may be designed to give an **orthonormal perfect reconstruction** wavelet transform.
- Tree **b** filters are the **reverse** of tree **a** filters, and reconstruction filters are the reverse of analysis filters, so **all filters** are from the **same orthonormal set**.
- Both trees have the **same frequency responses**.
- **Symmetric sub-sampling** – see below.

# Q-shift DT CWT Basis Functions – Levels 1 to 3

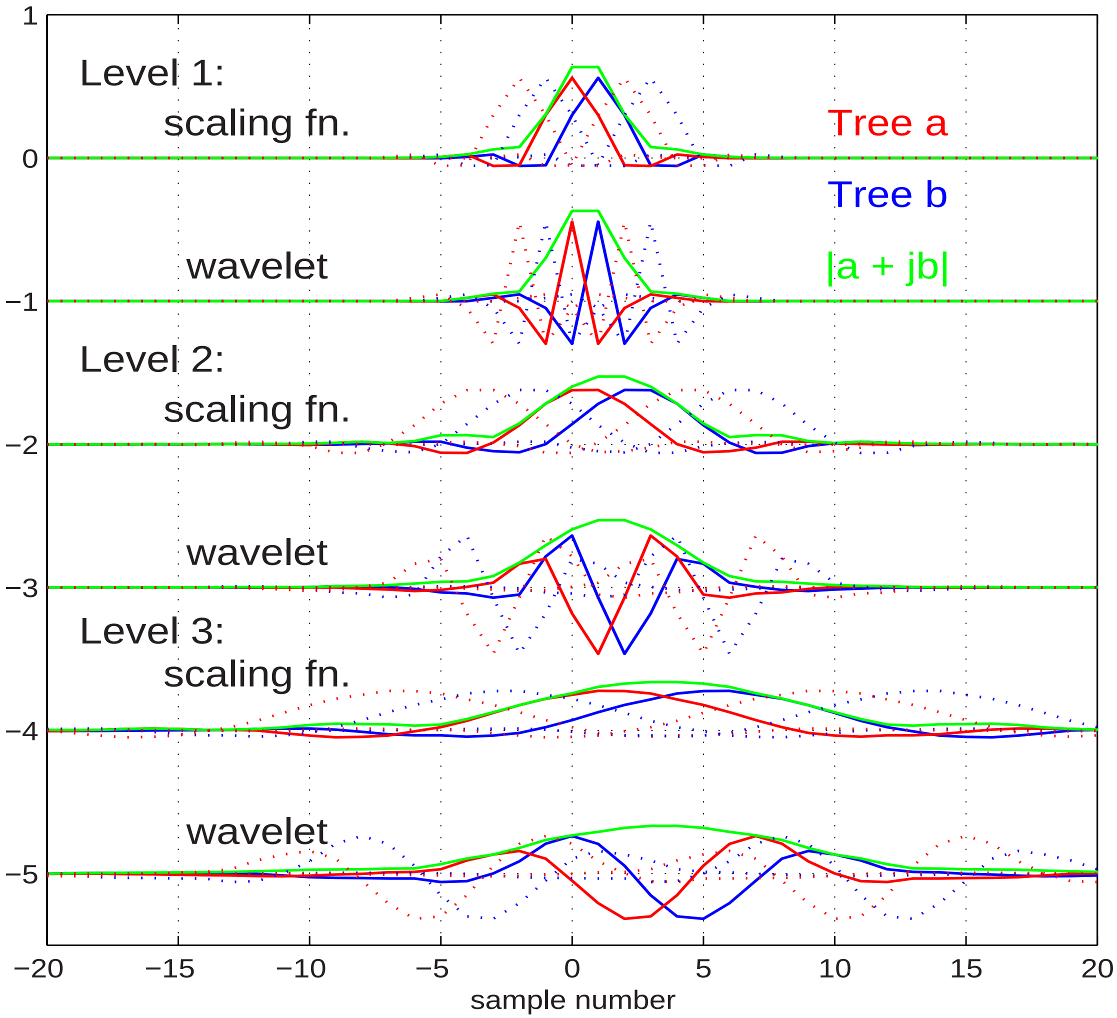



Figure 3: Basis functions for adjacent sampling points are shown dotted.

## Sampling Symmetries

In a regular multi-resolution pyramid structure each parent coefficient must lie symmetrically below the mean position of its 2 children (4 children in 2-D). Each filter should also be symmetric about its mid point.

- For the **Q-shift filters**, fig 3 shows that parents are symmetrically below their children, and that Hi and Lo filters at each level are aligned correctly.

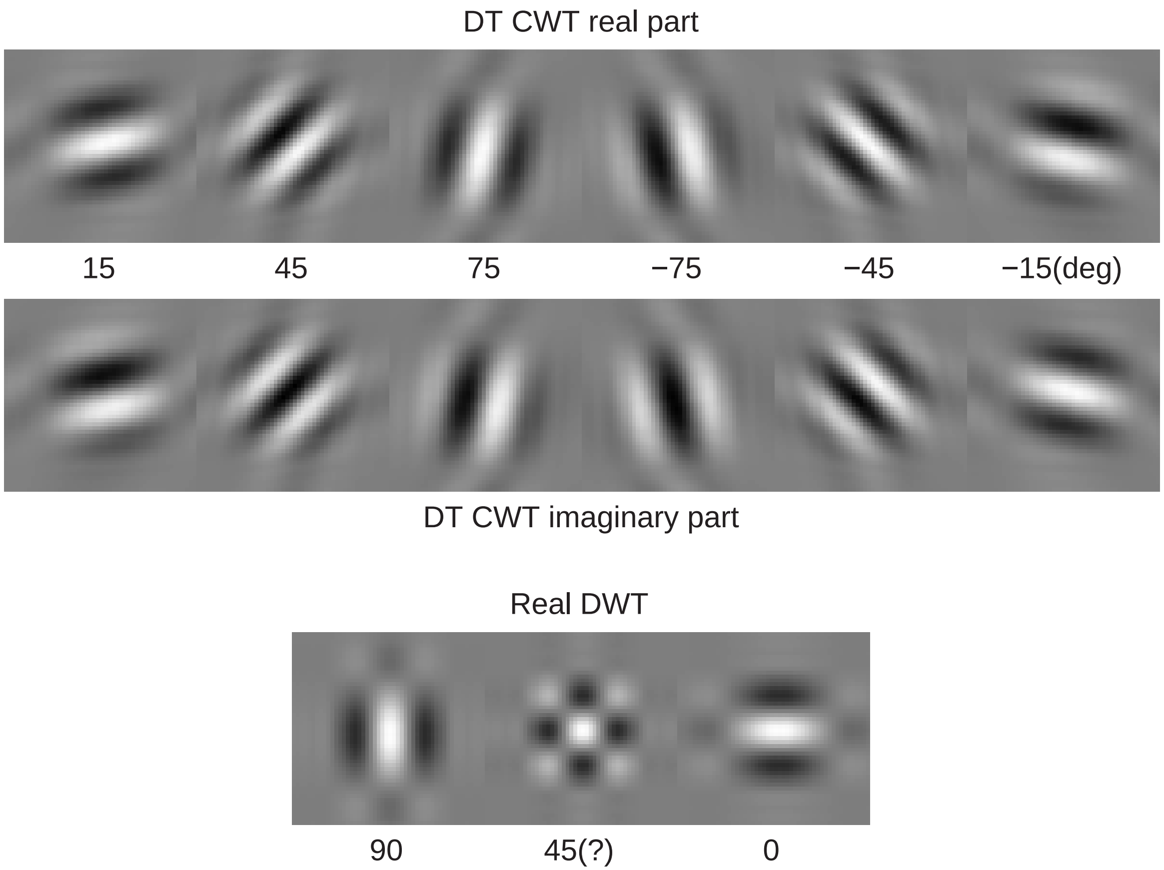
- Since one Q-shift tree is the time-reverse of the other, the combined **complex** impulse responses are **conjugate symmetric** about their mid points, even though the separate responses are asymmetric (see fig 3, right). Hence **symmetric extension** is still an effective technique at image edges.

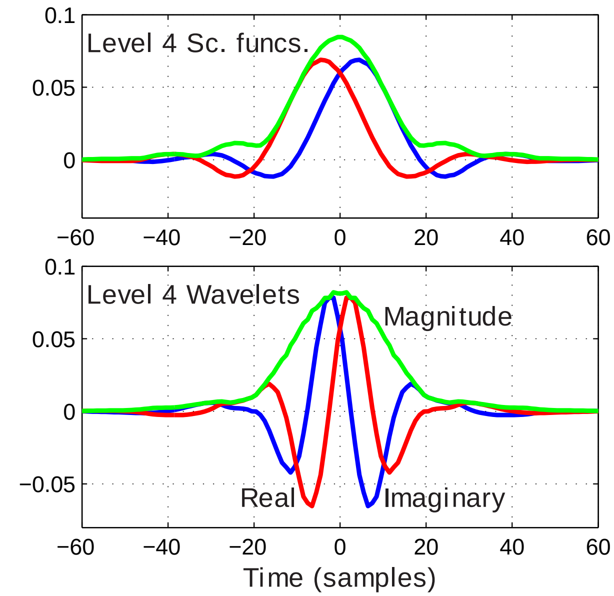
## The DT CWT in 2-D

When the DT CWT is applied to 2-D signals (images), it has the following features:

- It is performed separably, with 2 trees used for the rows of the image and 2 trees for the columns – yielding a **Quad-Tree** structure (4:1 redundancy).
- The 4 quad-tree components of each coefficient are combined by simple sum and difference operations to yield a **pair of complex coefficients**. These are part of two separate subbands in adjacent quadrants of the 2-D spectrum.
- This produces **6 directionally selective subbands** at each level of the 2-D DT CWT. Fig 4 shows the basis functions of these subbands at level 4, and compares them with the 3 subbands of a 2-D DWT.
- The DT CWT is directionally selective (see fig 7) because the complex filters can **separate positive and negative frequency components** in 1-D, and hence **separate adjacent quadrants** of the 2-D spectrum. Real separable filters cannot do this!

# 2-D Basis Functions at level 4




$$e^{j\omega_1 x}\ e^{j\omega_2 y} = e^{j(\omega_1 x + \omega_2 y)}$$

Figure 4: Basis functions of 2-D Q-shift complex wavelets (top), and of 2-D real wavelet filters (bottom), all illustrated at level 4 of the transforms. The complex wavelets provide 6 directionally selective filters, while real wavelets provide 3 filters, only two of which have a dominant direction. The 1-D bases, from which the 2-D complex bases are derived, are shown to the right.

# 2 LEVELS OF DT CWT IN 2-D

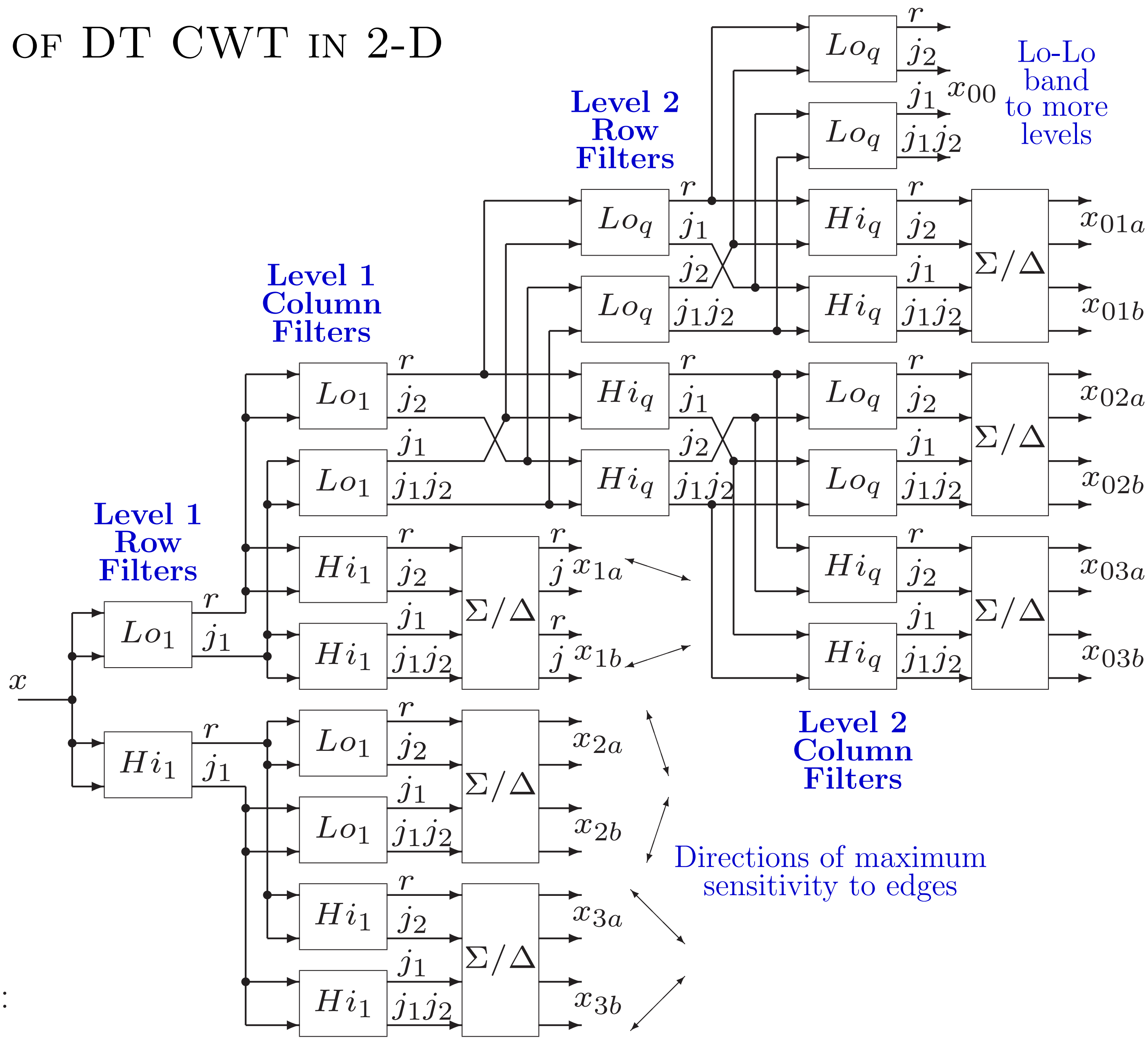
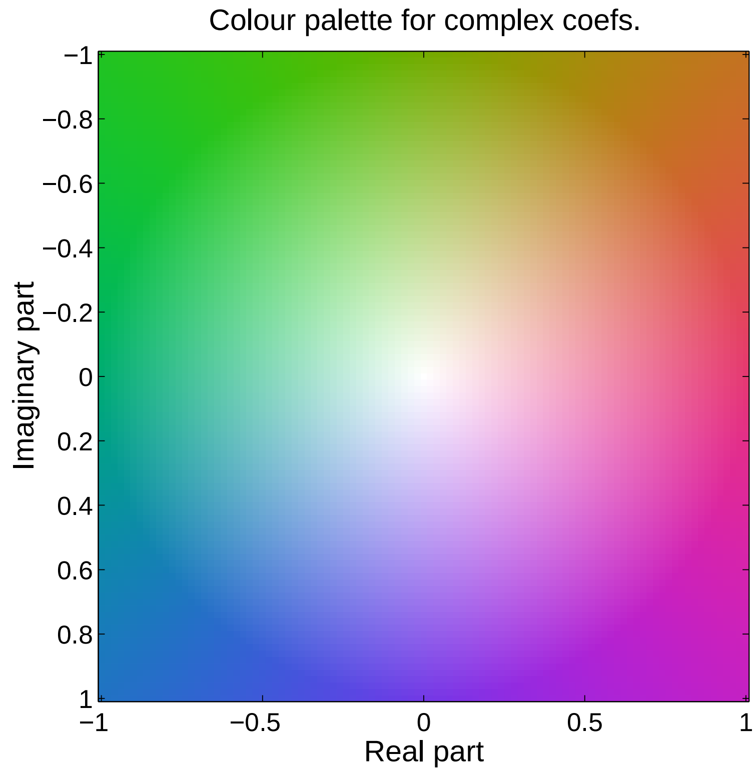



Figure 5:

# Test Image and Colour Palette for Complex Coefficients



# 2-D DT-CWT Decomposition into Subbands

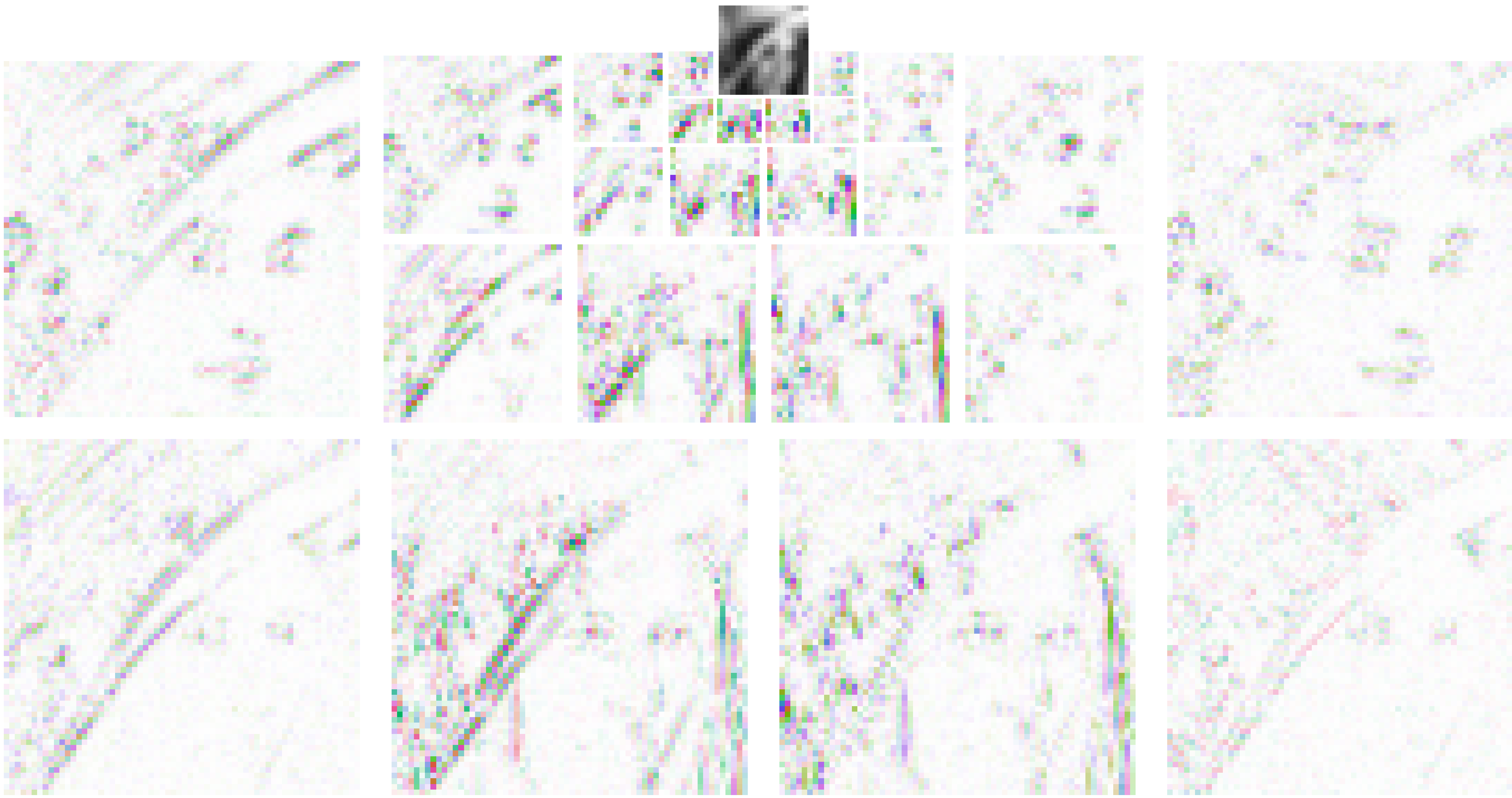


Figure 6: Four-level DT-CWT decomposition of *Lenna* into 6 subbands per level (only the central $128 \times 128$ portion of the image is shown for clarity). A colour-wheel palette is used to display the complex wavelet coefficients.

# 2-D DT-CWT Reconstruction Components from Each Subband

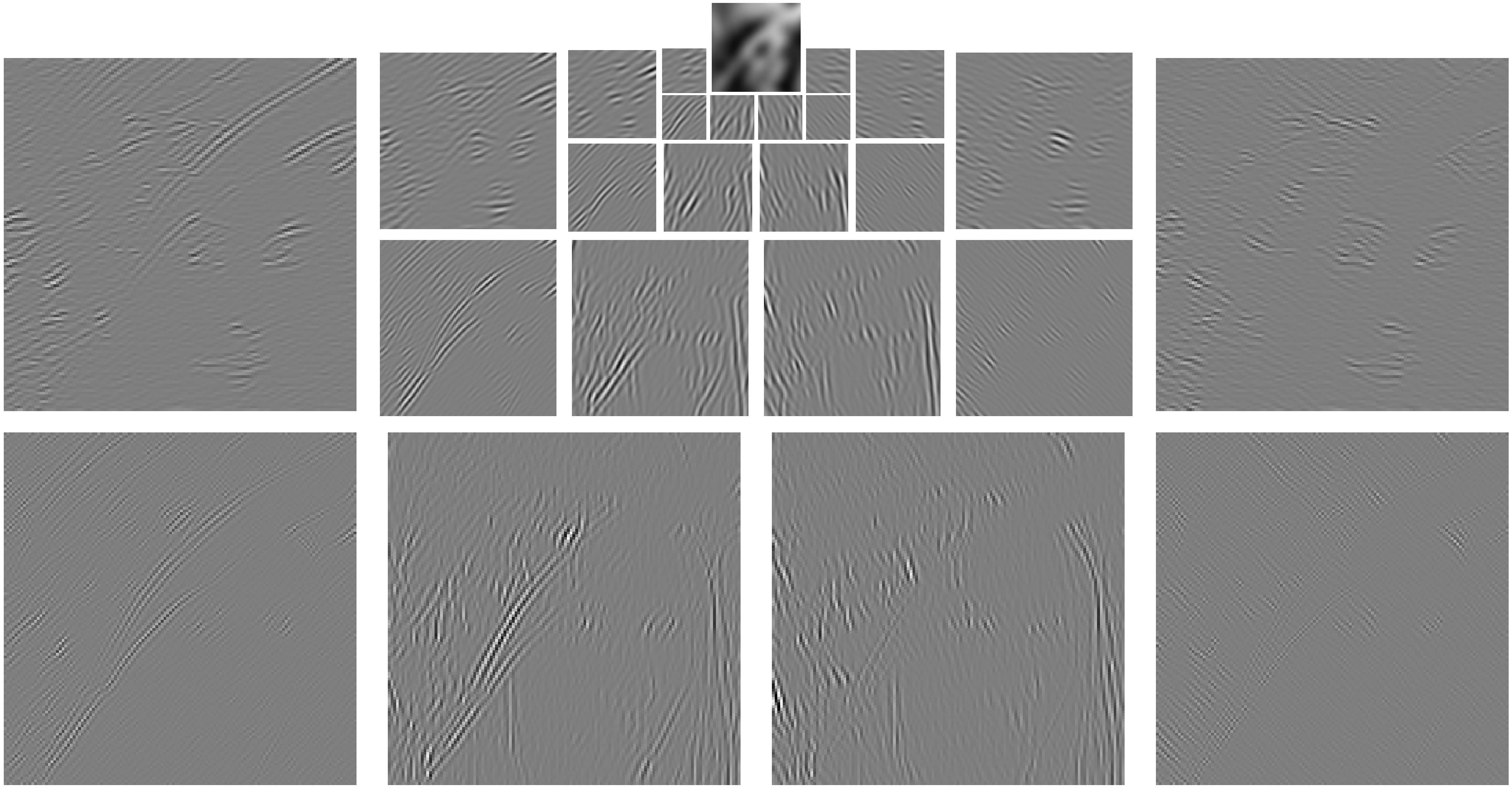


Figure 7: Components from each subband of the reconstructed output image for a 4-level DT-CWT decomposition of *Lenna* (central $128 \times 128$ portion only).

# 2-D Shift Invariance of DT CWT vs DWT

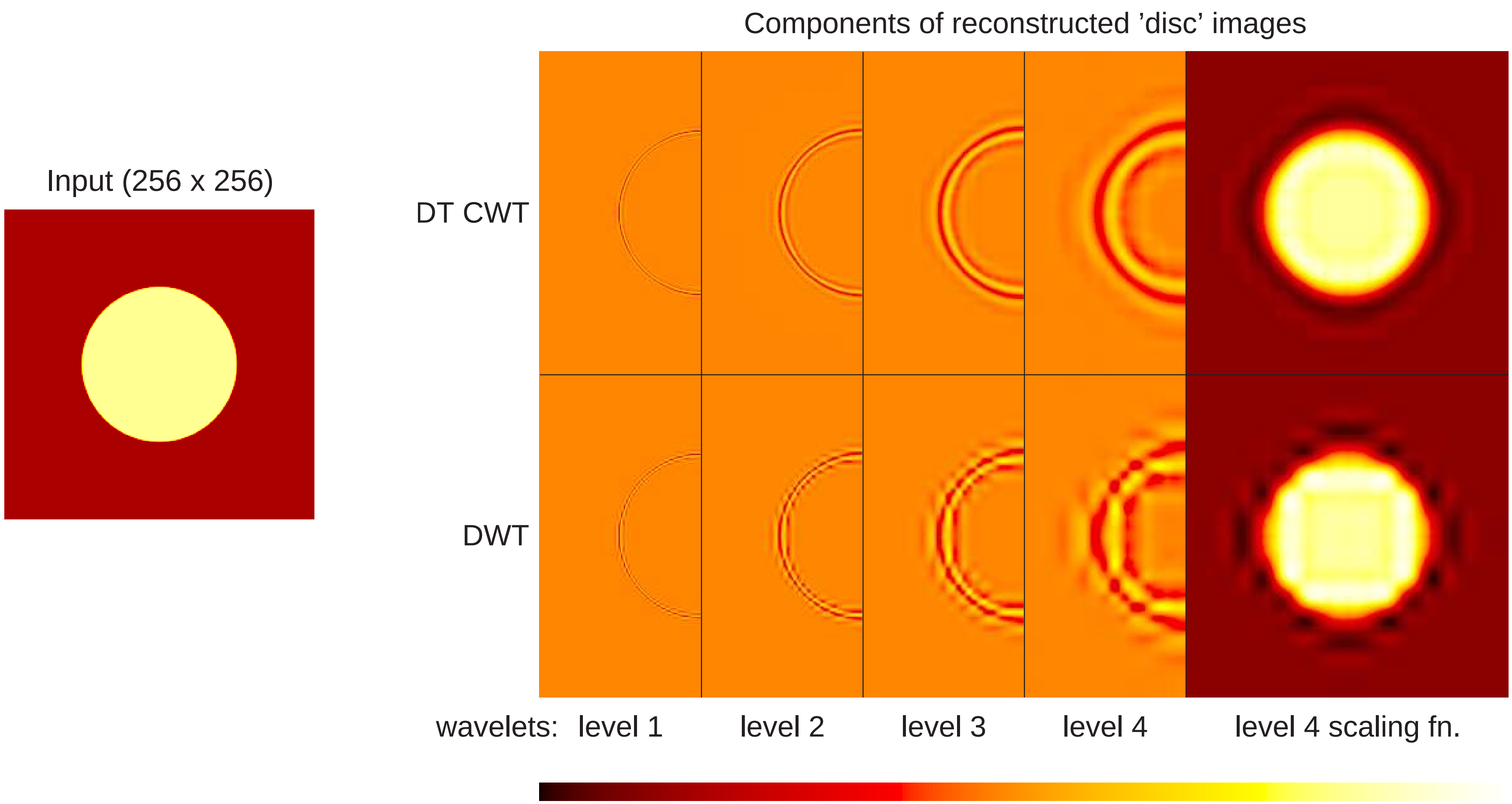



Figure 8: Wavelet and scaling function components at levels 1 to 4 of an image of a light circular disc on a dark background, using the 2-D DT CWT (upper row) and 2-D DWT (lower row). Only half of each wavelet image is shown in order to save space.

## The DT CWT in 3-D

When the DT CWT is applied to 3-D signals (eg medical MRI or CT datasets), it has the following features:

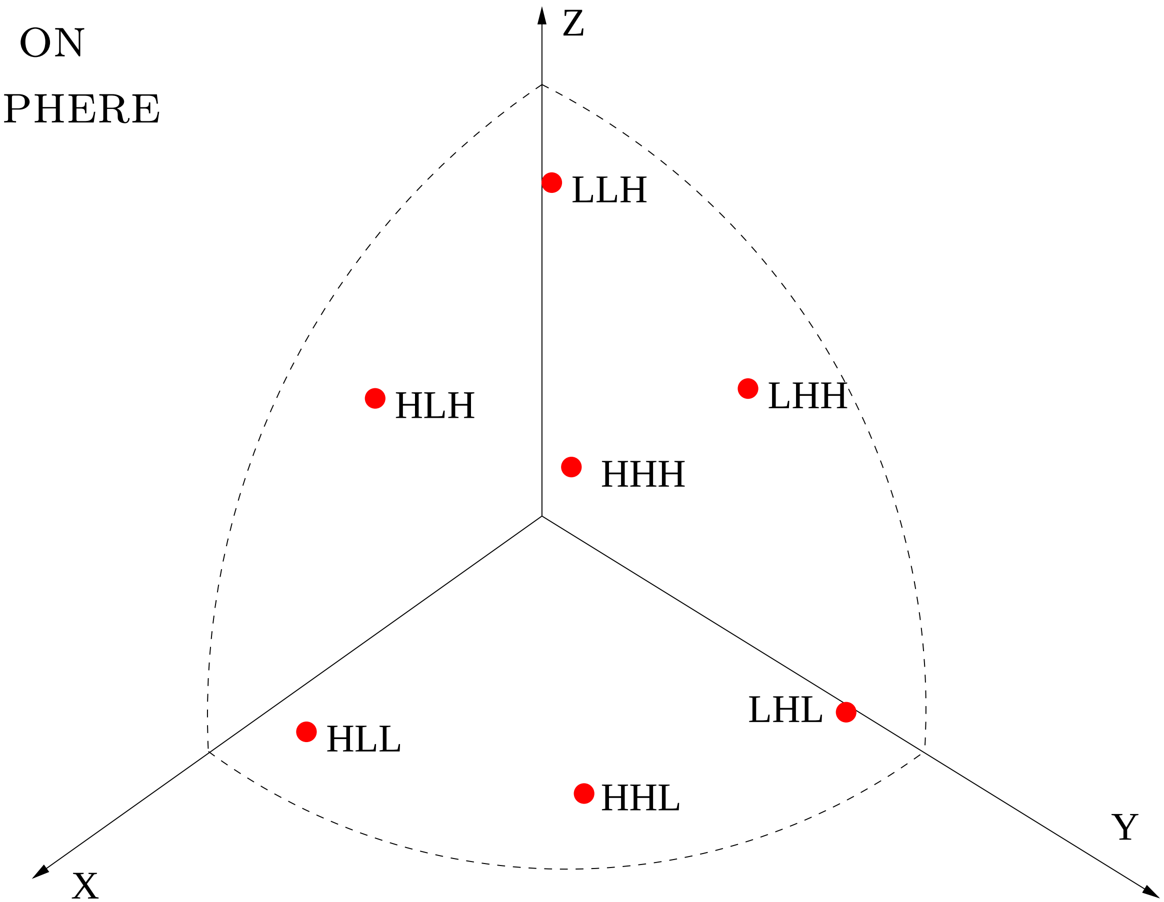
- It is performed separably, with 2 trees used for the rows, 2 trees for the columns and 2 trees for the slices of the 3-D dataset – yielding an **Octal-Tree** structure (8:1 redundancy).

- The 8 octal-tree components of each coefficient are combined by simple sum and difference operations to yield a **quad of complex coefficients**. These are part of 4 separate subbands in adjacent octants of the 3-D spectrum.

- This produces **28 directionally selective subbands** $(4 \times 8 - 4)$ at each level of the 3-D DT CWT. The subband basis functions are now **planar waves** of the form $e^{j(\omega_1 x + \omega_2 y + \omega_3 z)}$, modulated by a 3-D Gaussian envelope.

- Each subband responds to approximately flat surfaces of a particular orientation. There are 7 orientations on each quadrant of a hemisphere.

# 3D subband orientations on one quadrant of a hemisphere




3D frequency domain:

3D Gabor-like basis functions:

$$h_{k1,k2,k3}(x,y,z) \simeq e^{-(x^2+y^2+z^2)/2\sigma^2} \times e^{j(\omega_{k1}\,x + \omega_{k2}\,y + \omega_{k3}\,z)}$$

These are **28 planar waves** (7 per quadrant of a hemisphere) whose orientation depends on $\omega_{k1} \in \{\omega_L, \omega_H\}$ and $\omega_{k2}, \omega_{k3} \in \{\pm\omega_L, \pm\omega_H\}$, where $\omega_H \simeq 3\omega_L$.

## Applications

The Q-shift DT CWT provides a valuable analysis and reconstruction tool for a variety of application areas:

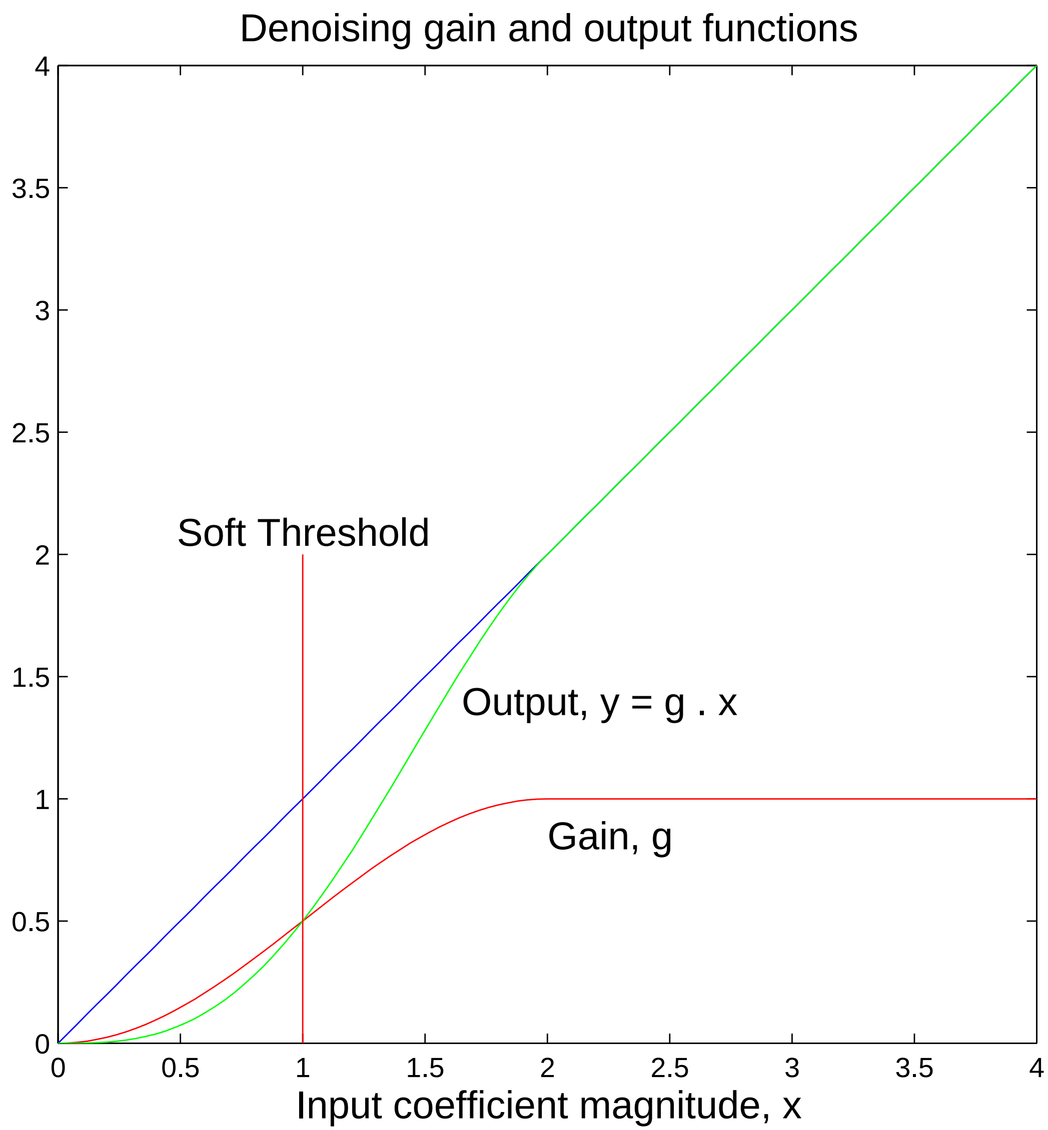
- **Motion estimation** [Magarey 98] and **compensation**
- **Registration** [Kingsbury 02]
- **Denoising** [Choi 00] and **deconvolution** [Jalobeanu 00, De Rivaz 01]
- **Texture analysis** [Hatipoglu 99] and **synthesis** [De Rivaz 00]
- **Segmentation** [De Rivaz 00, Shaffrey 02]
- **Classification** [Romberg 00] and **image retrieval** [Kam & Ng 00, Shaffrey 03]
- **Watermarking of images** [Loo 00] and **video** [Earl 03]
- **Compression / Coding** [Reeves 03]
- **Seismic analysis** [van Spaendonck & Fernandes 02]
- **Diffusion Tensor MRI visualisation** [Zymnis 04]

## De-Noising – Method:

- Transform the noisy input image to **compress the image energy** into as few coefs as possible, leaving the noise well distributed.
- Suppress lower energy coefs (mainly noise).
- Inverse transform to recover de-noised image.

## What is the Optimum Transform ?

- **DWT** is better than **DCT** or **DFT** for compressing image energy.
- But DWT is **shift dependent** – Is a coef small because there is no signal energy at that scale and location, **or** because it is sampled near a zero-crossing in the wavelet response?
- The **undecimated DWT** can solve this problem but at **significant cost** – redundancy (and computation) is increased by $3M : 1$, where $M$ is no. of DWT levels.
- The **DT CWT** has only $4 : 1$ redundancy, is directionally selective, and works well.


Denoising gain and output functions
Soft Threshold
Output, y = g . x
Gain, g
Input coefficient magnitude, x
0
0.5
1
1.5
2
2.5
3
3.5
4

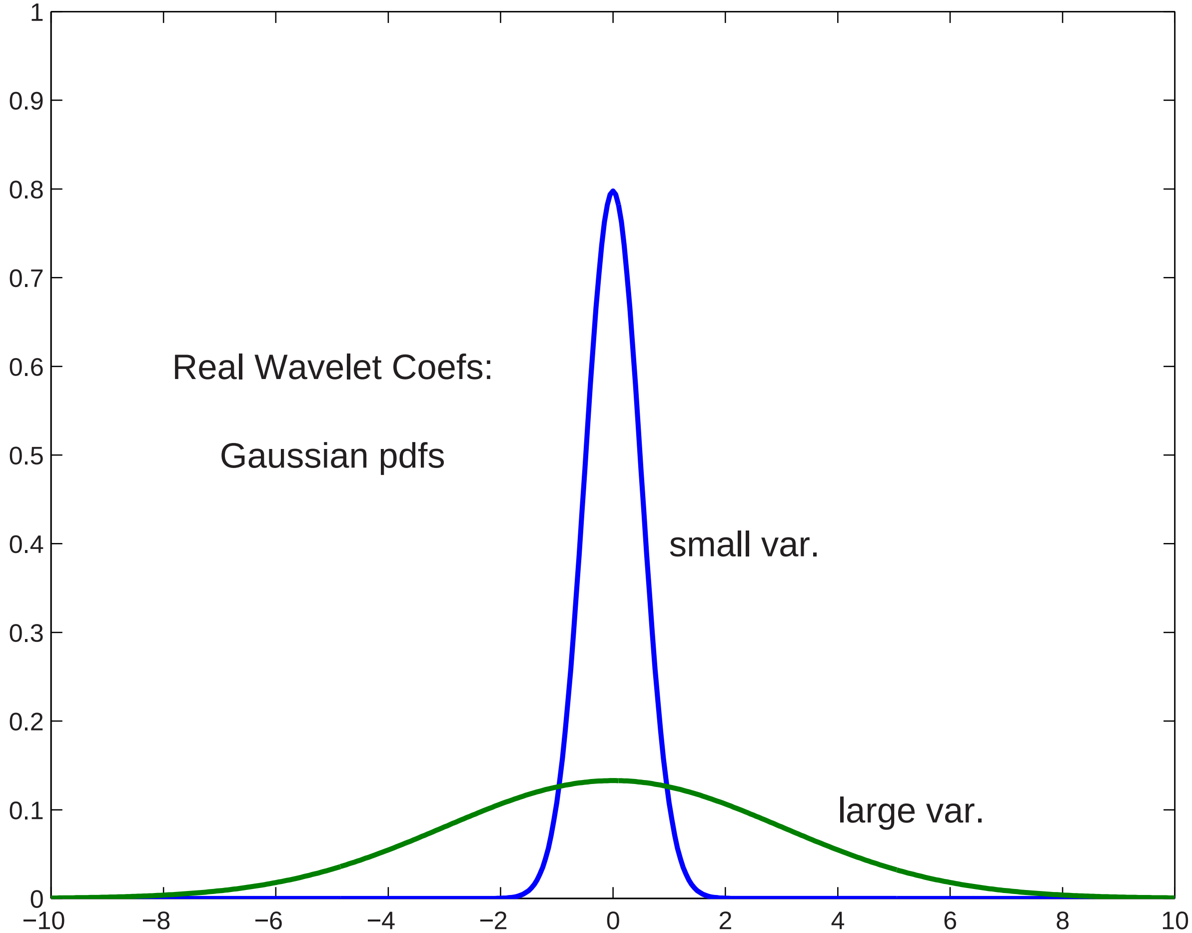



Figure 9: Probability density functions (pdfs) of small and large variance Gaussian distributions, typical for modelling **real and imaginary parts** of complex wavelet coefficients.

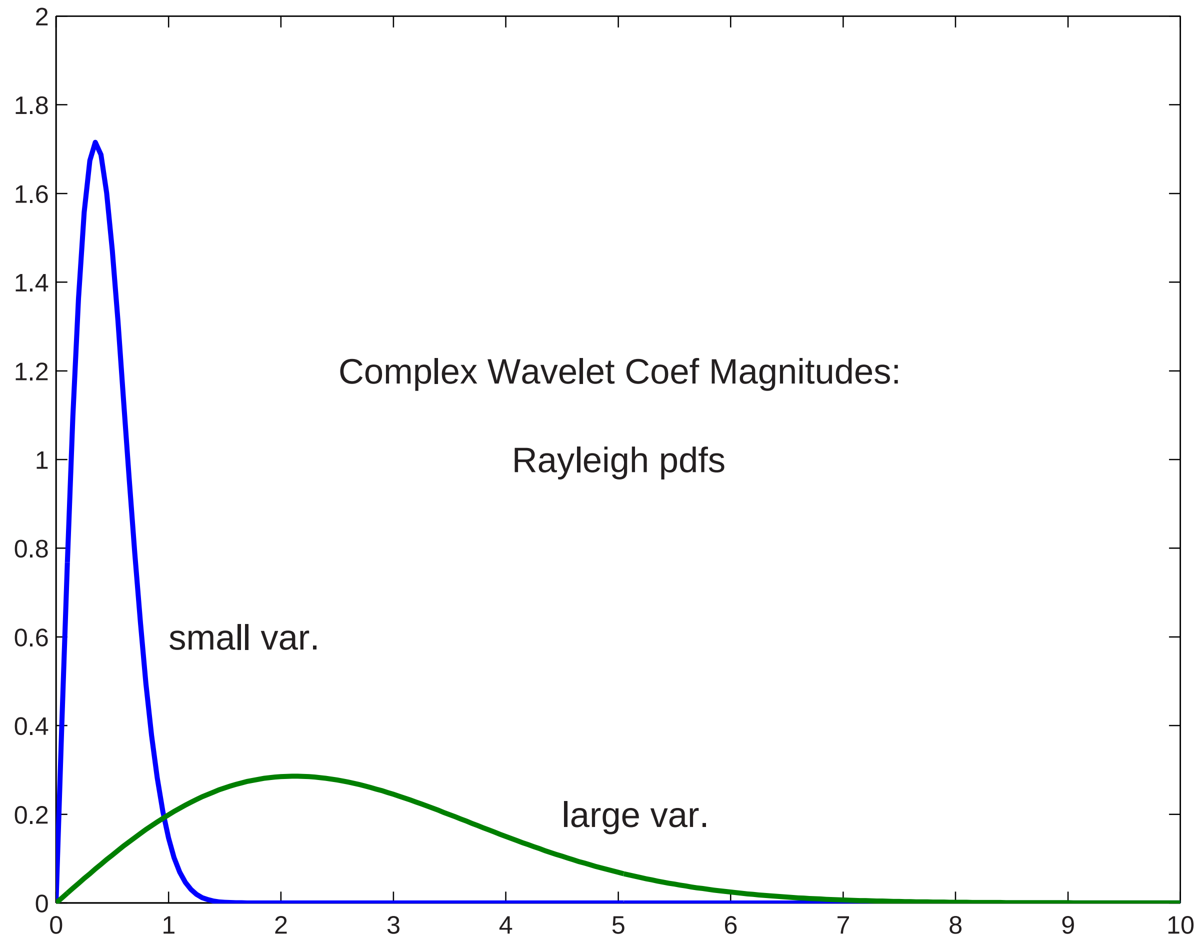



Figure 10: Probability density functions (pdfs) of small and large variance Rayleigh distributions, typical for modelling **magnitudes** of complex wavelet coefficients.

# Image Denoising with different Wavelet Transforms - Lenna

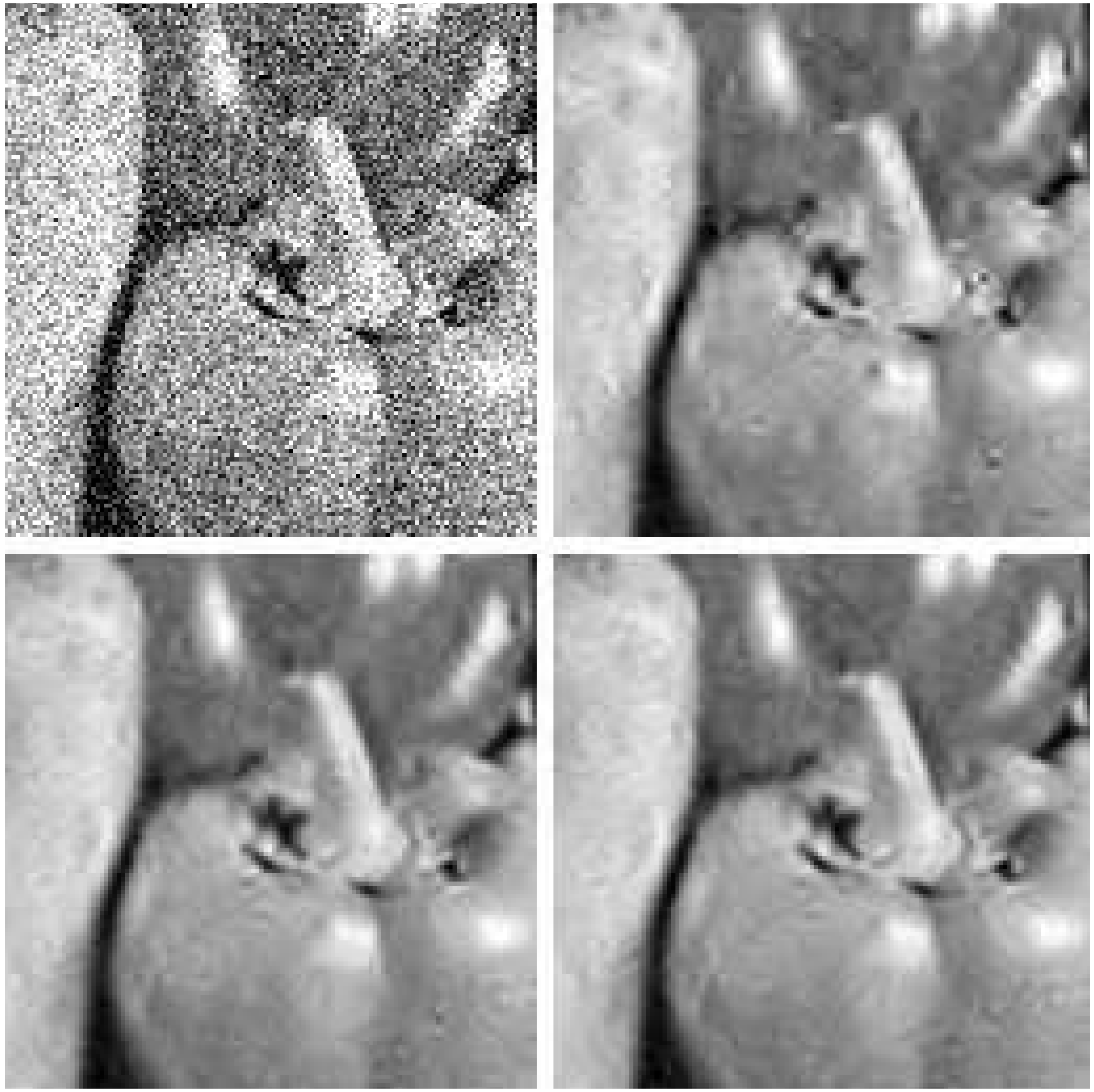
AWGN
SNR =
3.0 dB

Real DWT
SNR =
11.67 dB

Undec. WT
SNR =
12.82 dB

DT CWT
SNR =
12.99 dB

# Image Denoising with different Wavelet Transforms - Peppers



AWGN
SNR =
3.0 dB

Real DWT
SNR =
12.24 dB

Undec. WT
SNR =
13.45 dB

DT CWT
SNR =
13.51 dB

# HEIRARCHICAL DENOISING WITH GAUSSIAN SCALE MIXTURES (GSMs)

Non-heir.
DT CWT
SNR =
12.99 dB

Heirarchical
DT CWT
SNR =
13.51 dB

Non-heir.
DT CWT
SNR =
13.51 dB

Heirarchical
DT CWT
SNR =
13.85 dB

## Denoising a 3-D dataset

e.g. Medical 3-D MRI or helical CT scans.

**Method:**

- Perform 3-D DT CWT on the dataset.
- Attenuate smaller coefficients, based on their magnitudes, as for 2-D denoising. (Heirarchical methods are also quite feasible.)
- Perform inverse 3-D DT CWT to recover the denoised dataset.

A Matlab example shows denoising of an ellipsoidal surface, buried in Gaussian white noise.

## Conclusions – Part 1

The **Dual-Tree Complex Wavelet Transform** provides:

- Approximate **shift invariance**
- **Directionally selective** filtering in 2 or more dimensions
- **Low redundancy** – only $2^m : 1$ for $m$-D signals
- **Perfect reconstruction**
- **Orthonormal filters** below level 1, but still giving **linear phase** (conjugate symmetric) complex wavelets
- **Low computation** – order-$N$; less than $2^m$ times that of the fully decimated DWT ($\sim 3.3$ times in 2-D, $\sim 5.1$ times in 3-D)

## Conclusions (cont.)

- A **general purpose multi-resolution front-end** for many image analysis and reconstruction tasks:
  - Enhancement (deconvolution)
  - Denoising
  - Motion / displacement estimation and compensation
  - Texture analysis / synthesis
  - Segmentation and classification
  - Watermarking
  - 3D data enhancement and visualisation
  - Coding (?)

Papers on complex wavelets are available at:

**http://www.eng.cam.ac.uk/~ngk/**

A Matlab DTCWT toolbox is available on request from:

**ngk@eng.cam.ac.uk**