STATISTICAL IMAGE MODELLING USING INTERSCALE PHASE RELATIONSHIPS OF COMPLEX WAVELET COEFFICIENTS

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ABSTRACT

A novel method for modelling the statistics of 2D photographic images useful in image restoration is defined. The new method is based on the Dual Tree Complex Wavelet Transform (DT-CWT) but a phase rotation is applied to the coefficients to create complex coefficients whose phase is shift-invariant at multiscale edge and ridge features. This is in addition to the magnitude shift invariance achieved by the DT-CWT. The increased correlation between coefficients adjacent in space and scale provides an improved mechanism for signal estimation.

1. INTRODUCTION

The set of natural images is a small fraction of the space of all possible images. Modelling the statistics of natural signals is a challenging task partly due to the high dimensionality of the signal. To make progress, it is essential to reduce the dimensionality of the space on which one defines the probability model. Certain assumptions are commonly made about the signal in order to simplify matters.

The sparseness property of wavelet coefficients and tendency of wavelets bases to diagonalise images allows us to break the problem into modelling a small number of 'neighbouring' coefficients (in space and scale) to reduce the dimensionality and improve the tractability of the problem.

State-of-the-art wavelet image restoration methods are based on two major properties particular to image wavelet coefficients:

1. **Sparsity:** Energy tends to be concentrated in a small number of large coefficients with most coefficients being small.

2. **Clustering:** Coefficients display a self-reinforcing property where large-magnitude coefficients cluster in spatial location and in frequency (the latter is also known as persistence across scale).

We introduce a **new third property** of complex wavelet coefficients for image denoising:

3. **Interscale phase consistency:** The phases of wavelet coefficients display a consistent relationship between coefficients at adjacent scales near image discontinuities, such as the edges of objects.

Methods of parameterising image statistics, including those of neighbourhoods of wavelet coefficients, also commonly involve the use of covariance information [2]. Here, we examine the covariance information for groups of coefficients near discontinuities and show how the information may be better represented using coefficients whose phase is adjusted using phase information from coefficients at the next coarser level.

The method is derived from the principle that the spectrum of an edge or ridge feature behaves similarly across a range of scales. The technique also makes use of the tendency of images to have significantly greater spectral content at lower frequencies and the resulting ability of multiscale denoising algorithms to achieve higher SNR performance at coarser scales.

In the following we use the q-shift version of the DT-CWT [1]. Coefficients in the subband at the next coarser level are referred to as parent coefficients. In the case of the DT-CWT, these need to be interpolated.

Although the method may be applicable to other image modelling applications, we develop the framework with signal estimation and separation applications in mind. The Wiener filtering example given in section §3 is chosen as the simplest way to illustrate the advantages of the interscale phase approach. We have also extended the new modelling to a more advanced multi-scale Gaussian Scale Mixture (GSM) denoising algorithm similar to that in [2]. Preliminary results are promising with improvement achieved visually and numerically at image discontinuities. However the resulting algorithm is beyond the scope of this paper and will be subject of a journal submission in the near future.

2. INTERSCALE PHASE RELATIONSHIPS

2.1. Background

A well known property of the Fourier Transform is that a shift in the time or spatial domain corresponds to a linear phase ramp in the Fourier domain.

$$h(t-\alpha) \rightleftharpoons H(f)e^{-i2\pi\alpha f} \tag{1}$$

Consider an input signal consisting of a particular object in the spatial or time domain. A shift of α will result in a phase shift of $2\pi\alpha f_0$ at frequency f_0 in the Fourier domain. A 'moving' object will cause the Fourier coefficients to rotate at a rate proportional to their frequency.

DT-CWT coefficients display similar properties to Fourier coefficients for small shifts of the dominant feature in the vicinity of the coefficient. DT-CWT subbands are centred on a frequencies exactly twice those of the next coarser level. Assume the presence of a single ridge or edge feature at a given scale, orientation and location. Because adjacent wavelet coefficients are at different locations relative to the feature, the phase of a complex coefficient will tend to be offset from its neighbour by an amount twice that of the corresponding parent coefficient interpolated at the same location as the child, provided the feature is multiscale and the frequency spectrum of the feature behaves similarly across both of the scales.

2.2. Motivation

Anderson *et al.* illustrated that the phase response to a step input is approximately linear for a 1-D wavelet transform near the main support of the wavelet basis function [3]. A similar result can be obtained for an impulse input. In both cases the phase of the child coefficient changes at approximately twice the rate of its parent.

The constancy of this phase gradient relationship near discontinuities motivates the definition of a new modified complex wavelet coefficient 'derotated' by twice the phase of the interpolated parent coefficient. In the presence of a multiscale feature the phases of these derotated coefficients should be approximately aligned and therefore highly correlated at major edge features. This correlation can then be used to provide improved modelling at image features.

2.3. Previous uses of interscale phase

Interscale phase relationships of wavelet coefficients have previously been used in other signal processing areas including object recognition [3] and texture synthesis [4]. Romberg *et al.* [5] discuss interscale phase relationships and it is closely related to the use of phase congruence in edge detection [6]. In both [3] and [4] the interscale phase relationships are captured using the modified product of coefficients at adjacent scales shown in equation (2), where x is a wavelet coefficient at a given scale and orientation and x_p is the corresponding interpolated parent coefficient at the same location.

$$\begin{aligned} |p| &= |x||x_p| \\ \angle(p) &= \angle(x) - 2\angle(x_p) \end{aligned} \tag{2}$$

2.4. Derotated complex wavelet coefficients

In some applications, including Bayesian estimation contexts, avoidance of non-linearity is important for mathematical tractability so that the transformation can be readily used in an estimation algorithm. For this reason we opt to preserve the magnitude at a given scale and simply derotate the child by twice the phase of the parent. The magnitude of the coefficient is preserved, thus avoiding the non-linearity of taking a product. Equation (3) defines the new derotated coefficient.

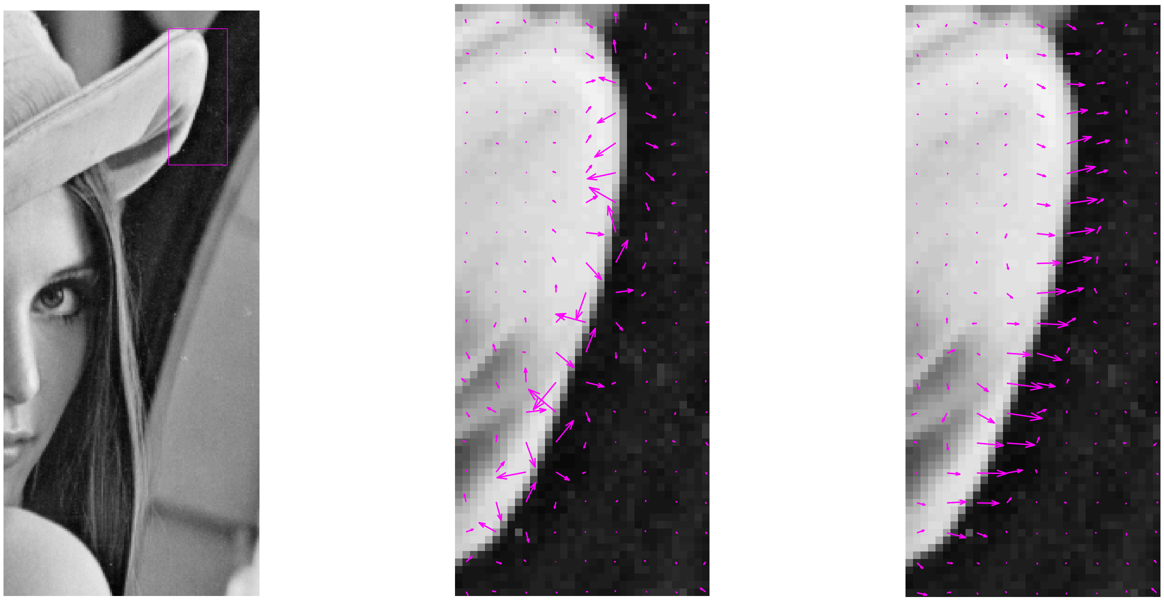
$$\begin{aligned} |q| &= |x| \\ \angle(q) &= \angle(x) - 2\angle(x_p) \end{aligned} \tag{3}$$

Interpolation of the parent level coefficients to obtain x_p is achieved using bandpass interpolation. This involves frequency shifting the subband to be centred on the origin of the frequency plane, performing standard 2D interpolation then reversing of the shift.

Figure 1 illustrates the alignment and improved correlation of the derotated coefficients as defined in (3) at an image edge.

Note that vectors not being aligned does not mean the coefficients are not correlated. Complex wavelet coefficients are a bandpass signal and will therefore rotate with an average rate that scales proportionally with the frequency of the passband. Note that covariance information can adequately describe a constant phase difference between coefficients across the image. However, for standard wavelet coefficients the phase differences between coefficients are governed by the particular characteristics of the dominant feature in a given area. For example, wavelet coefficients will rotate more slowly across a multiscale ridge than a edge because the spectrum has a lower 'centre of mass'. Also, the exact angle of the dominant feature impacts on the rate of rotation of vectors in a given area making the phase relationships inconsistent across the image. As seen in figure 1 the phase of the derotated coefficients is invariant to the exact feature angle within the angular support of the subband.

The increased correlation provides a new mechanism for signal prediction. An explanation of the mechanism in the case of image denoising is as follows. In the presence of noise the complex coefficients will be rotated from their correct positions. However, we know that the phases of the new coefficients should be aligned at discontinuities. Hence we can use an average of the phases in a small area to better predict the central one. In reality, we can use a probabilistic framework to perform the estimation in an optimal manner but it is based on this principle.



(a) Image (b) Standard complex wavelet coefficients (c) Derotated coefficients

Fig. 1. New property of wavelet coefficients for image restoration: interscale phase consistency at discontinuities.

3. WIENER FILTERING

To demonstrate the effect of improved covariance information in signal estimation and to demonstrate the advantages of modelling using interscale relationships we show the effect of performing Wiener denoising of neighbourhoods of derotated wavelet coefficients compared to using standard wavelet coefficients. Wiener filtering is chosen for the demonstration for its simplicity and its potential for use in more advanced algorithms such as that in [2].

The denoising problem is stated in equation (4), where $\mathbf{y}_s$ and $\mathbf{x}_s$ are the noisy and clean images.

$$\mathbf{y}_s = \mathbf{x}_s + \mathbf{n}_s \tag{4}$$

Transforming (4) to the wavelet domain, we use equation (5) to describe a neighbourhood of observed wavelet coefficients $\mathbf{y}$. $\mathbf{x}$ and $\mathbf{n}$ represent the wavelet coefficients of the image and noise.

$$\mathbf{y} = \mathbf{x} + \mathbf{n} \tag{5}$$

To model the signal using derotated coefficients we use equation (6). $\mathbf{A}$ is a unitary rotation matrix, which converts a set of derotated coefficients $\mathbf{q}$ to the corresponding DT-CWT coefficients using the phase of the interpolated parent coefficients. Based on the assumption that an edge or ridge feature of a given polarity is equally likely to one of the opposite polarity, $\mathbf{q}$ is assumed to have zero mean.

$$\mathbf{x} = \mathbf{A}\,\mathbf{q} \tag{6}$$

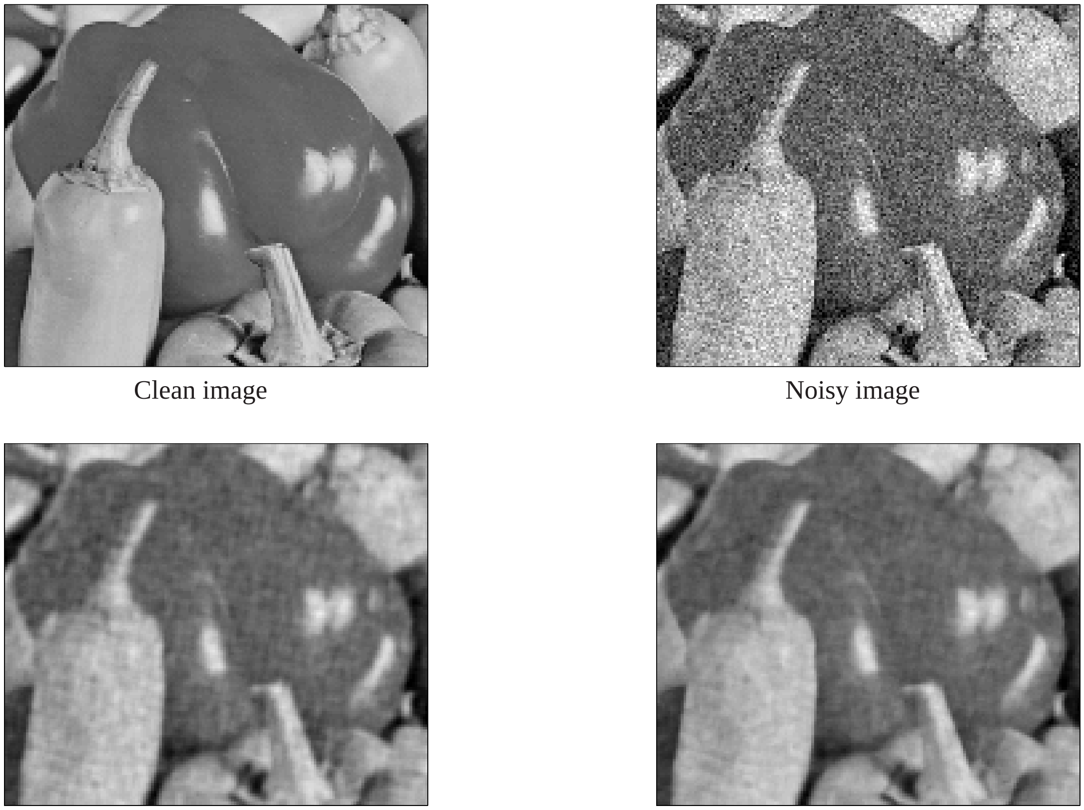
Wiener estimation of a neighbourhood of wavelet coefficients from the observed coefficients $\mathbf{y}$ is given in equation (7). It assumes both the noise and signal are Gaussian with covariance $\mathbf{C_n}$ and $\mathbf{C_m}$.

$$E\{\mathbf{x}|\mathbf{y}\} = \mathbf{C_m}\left(\mathbf{C_m} + \mathbf{C_n}\right)^{-1}\mathbf{y} \tag{7}$$

When using standard wavelet coefficients, $\mathbf{C_m}$ is the covariance matrix $\mathbf{C_x}$ of the neighbourhood $\mathbf{x}$. When employing derotated coefficients, $\mathbf{C_m}$ becomes $\mathbf{A}\mathbf{C_q}\mathbf{A}^T$ where $\mathbf{C_q}$ is the covariance of $\mathbf{q}$. In this case the covariance becomes spatially adaptive and conditional on the phase of coefficients at the parent level. $\mathbf{C_x}$ and $\mathbf{C_q}$ are estimated from the noisy data based on the methods described in [2].

Figure 2 shows the results of Wiener denoising using a six coefficient neighbourhood including the parent and four directly adjacent coefficients. The noise is white with standard deviation 25. We have obtained the final image estimates by performing an inverse wavelet transform of the central wavelet coefficients in each neighbourhood. The parent level phases required for derotation are approximated using the noisy coefficients. This approach assumes that the spectral characteristics of typical noisy natural images have a significantly higher signal to noise ratio at the parent scale relative to that of the child. The reconstructions contain the dappled noise typical of stationary Wiener filtering. However, when using the derotated coefficients, the noise level is lower and edges are significantly less blurred.

The SNR improvement gained by using the derotated coefficients for the whole *Peppers* image is 0.49dB averaged over eight different noise sample sets. For the full algorithm using GSM modelling the improvement is 0.18dB.



Clean image | Noisy image

Wiener filtering of DT-CWT coefficients | Wiener filtering of derotated coefficients

Fig. 2. Denoising via wiener filtering of neighbourhoods of derotated DT-CWT coefficients compared to that using standard DT-CWT coefficients for the *Peppers* image.

4. CONCLUSIONS

Significant gains in noise reduction can be made by modelling the non-stationarity of images, i.e. that various image components display different behaviour, for example, smooth areas, areas of texture and discontinuities. State-of-the-art GSM denoising algorithms do an excellent job of using the clustering property of wavelet coefficients to model images according to the activity within neighbourhoods of wavelet coefficients and attenuating coefficients heavily in the ‘inactive’ regions to remove noise [2]. The use of interscale phase relationships affords modelling of the non-stationary nature of the coefficients’ phase and to specifically model edge and ridge discontinuities and other active areas.

5. REFERENCES

[1] N. G. Kingsbury, “Complex wavelets for shift invariant analysis and filtering of signals,” *Journal of Applied and Computational Harmonic Analysis*, vol. 10, no. 3, pp. 234–253, May 2001.

[2] J. Portilla, V. Strela, M. J. Wainwright, and E. P. Simoncelli, “Image denoising using Gaussian scale mixtures in the wavelet domain,” *IEEE Transactions on Image Processing*, vol. 12, no. 11, pp. 1338–1351, Nov 2003.

[3] R. Anderson, N. Kingsbury, and J. Fauqueur, “Coarse level object recognition using interlevel products of complex wavelets,” in *IEEE International Conference on Image Processing*, Sep 2005.

[4] J. Portilla and E. P. Simoncelli, “A parametric texture model based on joint statistics of complex wavelet coefficients,” *International Journal of Computer Vision*, vol. 40, no. 1, pp. 49–71, Dec 2000.

[5] Hyeokho Choi Justin K. Romberg and Richard G. Baraniuk, “Multiscale edge grammars for complex wavelet transforms,” in *IEEE International Conference on Image Processing*, Thessaloniki, Greece, Oct 2001, pp. 614–617.

[6] P. Kovesi, “Image features from phase congruency,” *Videre: Journal of Computer Vision Research*, vol. 1, no. 3, pp. 2–26, May 1999.