**Complex Wavelets:**
**What are they and what can they do?**

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Complex Wavelets: What are they and what can they do?

- Basic form of the DT CWT
- Shift invariance of subband transfer functions
- DT CWT in 2-D – directional selectivity
- DT CWT in 3-D
- Denoising
- Image Registration
- Accumulated maps for keypoint detection
- Rotation-invariant local feature matching
Features of the (Real) Discrete Wavelet Transform (DWT)

- **Good compression** of signal energy.

- **Perfect reconstruction** with short support filters.

- **No redundancy**.

- **Very low computation** – order-$N$ only.

But

- **Severe shift dependence**.

- **Poor directional selectivity** in 2-D, 3-D etc.

The DWT is normally implemented with a tree of highpass and lowpass filters, separated by $2 : 1$ decimators.
**Real Discrete Wavelet Transform (DWT) in 1-D**

Figure 1: (a) Tree of real filters for the DWT. (b) Reconstruction filter block for 2 bands at a time, used in the inverse transform.
Visualising Shift Invariance

- Apply a standard input (e.g. unit step) to the transform for a range of shift positions.
- Select the transform coefficients from just one wavelet level at a time.
- Inverse transform each set of selected coefficients.
- Plot the component of the reconstructed output for each shift position at each wavelet level.
- Check for shift invariance (similarity of waveforms).

See Matlab demonstration.
Features of the Dual Tree Complex Wavelet Transform (DT CWT)

• Good **shift invariance** = **negligible aliasing**. Hence transfer function through each subband is independent of shift **and** wavelet coefs can be interpolated within each subband, independent of all other subbands.

• Good **directional selectivity** in 2-D, 3-D etc. – derives from **analyticity** in 1-D (ability to separate positive from negative frequencies).

• **Perfect reconstruction** with short support filters.

• **Limited redundancy** – 2:1 in 1-D, 4:1 in 2-D etc.

• **Low computation** – much less than the undecimated (à trous) DWT.

Each tree contains purely real filters, but the two trees produce the **real and imaginary parts** respectively of each complex wavelet coefficient.
Q-shift Dual Tree Complex Wavelet Transform in 1-D

Figure 2: Dual tree of real filters for the Q-shift CWT, giving real and imaginary parts of complex coefficients from tree a and tree b respectively. Figures in brackets indicate the approximate delay for each filter, where $q = \frac{1}{4}$ sample period.
Features of the Q-shift Filters

Below level 1:

- Half-sample delay difference is obtained with filter delays of \( \frac{1}{4} \) and \( \frac{3}{4} \) of a sample period (instead of 0 and \( \frac{1}{2} \) a sample for our original DT CWT).

- This is achieved with an **asymmetric even-length** filter \( H(z) \) and its time reverse \( H(z^{-1}) \).

- Due to the asymmetry (like Daubechies filters), these may be designed to give an **orthonormal perfect reconstruction** wavelet transform.

- Tree b filters are the reverse of tree a filters, and reconstruction filters are the reverse of analysis filters, so **all filters** are from the **same orthonormal set**.

- Both trees have the **same frequency responses**.

- The combined **complex** impulse responses are **conjugate symmetric** about their mid points, even though the separate responses are asymmetric. Hence **symmetric extension** still works at image edges.
Q-shift DT CWT Basis Functions – Levels 1 to 3

Figure 3: Basis functions for adjacent sampling points are shown dotted.
Frequency Responses of 18-tap Q-shift Filters
Frequency Responses of 14-tap Q-shift filters

**Wavelets at level:** 4 3 2 1

**Scaling fn. at level 4**
Frequency Responses of 6-tap Q-shift Filters
The DT CWT in 2-D

When the DT CWT is applied to 2-D signals (images), it has the following features:

- It is performed separably, with 2 trees used for the rows of the image and 2 trees for the columns – yielding a Quad-Tree structure (4:1 redundancy).

- The 4 quad-tree components of each coefficient are combined by simple sum and difference operations to yield a pair of complex coefficients. These are part of two separate subbands in adjacent quadrants of the 2-D spectrum.

- This produces 6 directionally selective subbands at each level of the 2-D DT CWT. Fig 4 shows the basis functions of these subbands at level 4, and compares them with the 3 subbands of a 2-D DWT.

- The DT CWT is directionally selective (see fig 6) because the complex filters can separate positive and negative frequency components in 1-D, and hence separate adjacent quadrants of the 2-D spectrum. Real separable filters cannot do this!
2-D Basis Functions at Level 4

Figure 4: Basis functions of 2-D Q-shift complex wavelets (top), and of 2-D real wavelet filters (bottom), all illustrated at level 4 of the transforms. The complex wavelets provide 6 directionally selective filters, while real wavelets provide 3 filters, only two of which have a dominant direction. The 1-D bases, from which the 2-D complex bases are derived, are shown to the right.
Frequency Responses of 2-D Q-shift filters at levels 3 and 4

Contours shown at $-1 \text{ dB}$ and $-3 \text{ dB}$.
TEST IMAGE AND COLOUR PALETTE FOR COMPLEX COEFFICIENTS
2-D DT-CWT Decomposition into Subbands

Figure 5: Four-level DT-CWT decomposition of Lenna into 6 subbands per level (only the central $128 \times 128$ portion of the image is shown for clarity). A colour-wheel palette is used to display the complex wavelet coefficients.
2-D DT-CWT Reconstruction Components from Each Subband

Figure 6: Components from each subband of the reconstructed output image for a 4-level DT-CWT decomposition of Lenna (central $128 \times 128$ portion only).
2-D Shift Invariance of DT CWT vs DWT

Figure 7: Wavelet and scaling function components at levels 1 to 4 of an image of a light circular disc on a dark background, using the 2-D DT CWT (upper row) and 2-D DWT (lower row). Only half of each wavelet image is shown in order to save space.
THE DT CWT IN 3-D

When the DT CWT is applied to 3-D signals (eg medical MRI or CT datasets), it has the following features:

- It is performed separably, with 2 trees used for the rows, 2 trees for the columns and 2 trees for the slices of the 3-D dataset – yielding an Octal-Tree structure (8:1 redundancy).

- The 8 octal-tree components of each coefficient are combined by simple sum and difference operations to yield a quad of complex coefficients. These are part of 4 separate subbands in adjacent octants of the 3-D spectrum.

- This produces 28 directionally selective subbands \((4 \times 8 - 4)\) at each level of the 3-D DT CWT. The subband basis functions are now planar waves of the form \(e^{j(\omega_1 x + \omega_2 y + \omega_3 z)}\), modulated by a 3-D Gaussian envelope.

- Each subband responds to approximately flat surfaces of a particular orientation. There are 7 orientations on each quadrant of a hemisphere.
3D subband orientations on one quadrant of a hemisphere

3D frequency domain:

3D Gabor-like basis functions:

\[ h_{k_1, k_2, k_3}(x, y, z) \approx e^{-\frac{x^2 + y^2 + z^2}{2\sigma^2}} \times e^{j(\omega_{k_1} x + \omega_{k_2} y + \omega_{k_3} z)} \]

These are 28 planar waves (7 per quadrant of a hemisphere) whose orientation depends on \( \omega_{k_1} \in \{\omega_L, \omega_H\} \) and \( \omega_{k_2}, \omega_{k_3} \in \{\pm\omega_L, \pm\omega_H\} \), where \( \omega_H \simeq 3\omega_L \).
APPLICATIONS OF THE DT CWT

- **Motion estimation** [Magarey 98] and **compensation**
- **Registration** [Kingsbury 02]
- **Denoising** [Choi 00, Miller 06] and **deconvolution** [Jalobeanu 00, De Rivaz 01, J Ng 07]
- **Texture analysis** [Hatipoglu 99] and **synthesis** [De Rivaz 00]
- **Segmentation** [De Rivaz 00, Shaffrey 02]
- **Classification** [Romberg 00] and **image retrieval** [Kam & T T Ng 00, Shaffrey 03]
- **Watermarking of images** [Loo 00] and **video** [Earl 03]
- **Compression / Coding** [Reeves 03]
- **Seismic analysis** [van Spaendonck & Fernandes 02, Miller 05]
- **Diffusion Tensor MRI visualisation** [Zymnis 04]
- **Object matching & recognition** [Anderson & Fauqueur 06]
De-Noising – Method:

- Transform the noisy input image to **compress the image energy** into as few coefs as possible, leaving the noise well distributed.
- Suppress lower energy coefs (mainly noise).
- Inverse transform to recover de-noised image.

What is the Optimum Transform?

- **DWT** is better than **DCT** or **DFT** for compressing image energy.
- But DWT is **shift dependent** – Is a coef small because there is no signal energy at that scale and location, or because it is sampled near a zero-crossing in the wavelet response?
- The **undecimated DWT** can solve this problem but at **significant cost** – redundancy (and computation) is increased by $3M : 1$, where $M$ is no. of DWT levels.
- The **DT CWT** has only $4 : 1$ redundancy, is directionally selective, and works well.
Denoising gain and output functions

Input coefficient magnitude, $x$

Gain, $g$

Output, $y = g \cdot x$

Soft Threshold

Gain, $g$

Output, $y = g \cdot x$
Figure 8: Probability density functions (pdfs) of small and large variance Gaussian distributions, typical for modelling real and imaginary parts of complex wavelet coefficients.
Figure 9: Probability density functions (pdfs) of small and large variance Rayleigh distributions, typical for modelling magnitudes of complex wavelet coefficients.
Image Denoising with different Wavelet Transforms - Lenna

AWGN
SNR = 3.0 dB

Real DWT
SNR = 11.67 dB

Undec. WT
SNR = 12.82 dB

DT CWT
SNR = 12.99 dB
Image Denoising with different Wavelet Transforms - Peppers

AWGN
SNR = 3.0 dB

Real DWT
SNR = 12.24 dB

Undec. WT
SNR = 13.45 dB

DT CWT
SNR = 13.51 dB
Heirarchical Denoising with Gaussian Scale Mixtures (GSMs)

Non-heirarchical DT CWT
SNR = 12.99 dB

Heirarchical DT CWT
SNR = 13.51 dB

Non-heirarchical DT CWT
SNR = 13.51 dB

Heirarchical DT CWT
SNR = 13.85 dB
Denoising a 3-D dataset

e.g. Medical 3-D MRI or helical CT scans.

Method:

• Perform 3-D DT CWT on the dataset.

• Attenuate smaller coefficients, based on their magnitudes, as for 2-D denoising. (Hierarchical methods are also quite feasible.)

• Perform inverse 3-D DT CWT to recover the denoised dataset.

A Matlab example shows denoising of an ellipsoidal surface, buried in Gaussian white noise.
IMAGE REGISTRATION

KEY FEATURES OF ROBUST REGISTRATION ALGORITHMS

- Edge-based methods are more robust than point-based ones.
- Must be automatic (no human picking of correspondence points) in order to achieve sub-pixel accuracy in noise.
- Bandlimited multiscale (wavelet) methods will allow spatially adaptive denoising.
- Phase-based bandpass methods can give rapid convergence and immunity to illumination changes between images.
- Displacement field should be smooth, so use of a wide-area parametric (affine) model is preferable to local translation-only models.
Selected Method

- Dual-tree Complex Wavelet Transform (DT CWT):
  - provides complex coefficients whose phase shift depends approximately linearly with displacement;
  - allows each subband of coefficients to be interpolated independently of other subbands (because of shift invariance).

- Parametric model of displacement field, whose solution is based on local edge-based motion constraints (Hemmendorf et al., IEEE Trans Medical Imaging, Dec 2002):
  - derives straight-line constraints from directional subbands of DT CWT;
  - solves for model parameters which minimise constraint error energy over multiple directions and scales.
Image $A$  

DT CWT  

Select CWT levels according to iteration  

Shift within subbands  

Generate displacement field $\mathbf{v}(\mathbf{x}_i)$  

parameter field $\mathbf{a}_\mathcal{X}$  

Delay  

Inverse DT CWT  

Image $A$ registered to image $B$  

Image $B$  

DT CWT  

Form constraints $\mathbf{c}_i$  

Calculate $\mathbf{Q}_\mathcal{X}$ at each locality $\mathcal{X}$  

Smooth elements of $\mathbf{Q}_\mathcal{X}$ across image  

Solve for $\mathbf{a}_{\mathcal{X}, \text{min}}$ at each $\mathcal{X}$  

increment of parameter field $\mathbf{a}_\mathcal{X}$
Basic Linear Flow Model

Key Assumption for local translation model:

- Time derivative of the phase $\theta$ of each complex wavelet coefficient depends \textbf{approximately linearly} on the local velocity vector $\mathbf{v}$.

This can be expressed as a flow equation in time and spatial derivatives:

$$\frac{\partial \theta}{\partial t} = \nabla_x \theta \cdot \mathbf{v}$$

We can rearrange this to be in the form:

$$\nabla_x \theta \cdot \mathbf{v} - \frac{\partial \theta}{\partial t} = 0$$

or

$$\begin{bmatrix} \nabla_x \theta \\ -\frac{\partial \theta}{\partial t} \end{bmatrix}^T \tilde{\mathbf{v}} = 0 \quad \text{where} \quad \tilde{\mathbf{v}} = \begin{bmatrix} \mathbf{v} \\ 1 \end{bmatrix}$$
**Parametric Model: Constraint equations**

Let the displacement vector at the $i^{th}$ location $\mathbf{x}_i$ be $\mathbf{v}(\mathbf{x}_i)$; and let $\tilde{\mathbf{v}}_i = \begin{bmatrix} \mathbf{v}(\mathbf{x}_i) \\ 1 \end{bmatrix}$.

A straight-line constraint on $\mathbf{v}(\mathbf{x}_i)$ can be written

$$
\mathbf{c}_i^T \tilde{\mathbf{v}}_i = 0 \quad \text{or} \quad c_{1,i}v_{1,i} + c_{2,i}v_{2,i} + c_{3,i} = 0
$$

For a phase-based system in which wavelet coefficients at $\mathbf{x}_i$ in images $A$ and $B$ have phases $\theta_A$ and $\theta_B$, approximate phase linearity means that

$$
\mathbf{c}_i = C_i \begin{bmatrix} \nabla_{\mathbf{x}} \theta(\mathbf{x}_i) \\ \theta_B(\mathbf{x}_i) - \theta_A(\mathbf{x}_i) \end{bmatrix}
$$

In practise we compute this by averaging finite differences at the centre of a $2 \times 2 \times 2$ block of coefficients from images $A$ and $B$.

$C_i$ is a constant which does not affect the line defined by the constraint, but which is important later.
PARAMETERS OF THE MODEL

We can define an affine parametric model for $v$ such that

$$v(x) = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} a_3 & a_5 \\ a_4 & a_6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

or in a more useful form

$$v(x) = \begin{bmatrix} 1 & 0 & x_1 & 0 & x_2 & 0 \\ 0 & 1 & 0 & x_1 & 0 & x_2 \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_6 \end{bmatrix} = K(x) \cdot a$$

Affine models can synthesise translation, rotation, constant zoom, and shear.

A quadratic model, which allows for linearly changing zoom (approx perspective), requires up to 6 additional parameters and columns in $K$ of the form

$$\begin{bmatrix} \ldots & x_1x_2 & 0 & x_1^2 & 0 & x_2^2 & 0 \\ \ldots & 0 & x_1x_2 & 0 & x_1^2 & 0 & x_2^2 \end{bmatrix}$$
SOLVING FOR THE MODEL PARAMETERS

Let \( \tilde{K}_i = \begin{bmatrix} K(x_i) & 0 \\ 0 & 1 \end{bmatrix} \) and \( \tilde{a} = \begin{bmatrix} a \\ 1 \end{bmatrix} \) so that \( \tilde{v}_i = \tilde{K}_i \tilde{a} \).

Ideally for a given image locality \( \mathcal{X} \), we wish to find the parametric vector \( \tilde{a} \) such that

\[
\mathbf{c}_i^T \tilde{v}_i = 0 \quad \text{when} \quad \tilde{v}_i = \tilde{K}_i \tilde{a} \quad \text{for all } i \text{ such that } x_i \in \mathcal{X}.
\]

In practise this is an overdetermined set of equations, so we find the LMS solution, the value of \( a \) which minimises the squared error

\[
\mathcal{E}_\mathcal{X} = \sum_{i \in \mathcal{X}} \| \mathbf{c}_i^T \tilde{v}_i \|^2 = \sum_{i \in \mathcal{X}} \| \mathbf{c}_i^T \tilde{K}_i \tilde{a} \|^2 = \sum_{i \in \mathcal{X}} (\tilde{a}^T \tilde{K}_i^T \mathbf{c}_i)(\mathbf{c}_i^T \tilde{K}_i \tilde{a})
\]

\[
= \tilde{a}^T \tilde{Q}_\mathcal{X} \tilde{a} \quad \text{where} \quad \tilde{Q}_\mathcal{X} = \sum_{i \in \mathcal{X}} (\tilde{K}_i^T \mathbf{c}_i \mathbf{c}_i^T \tilde{K}_i)\]
SOLVING FOR THE MODEL PARAMETERS (cont.)

Since $\tilde{a} = \begin{bmatrix} a \\ 1 \end{bmatrix}$ and $\tilde{Q}_\mathcal{X}$ is symmetric, we define $\tilde{Q}_\mathcal{X} = \begin{bmatrix} Q & q \\ q^T & q_0 \end{bmatrix}$ so that

$$E_\mathcal{X} = \tilde{a}^T \tilde{Q}_\mathcal{X} \tilde{a} = a^T Q a + 2 a^T q + q_0$$

$E_\mathcal{X}$ is minimised when $\nabla_a E_\mathcal{X} = 2 Q a + 2 q = 0$, so $a_{\mathcal{X},\min} = - Q^{-1} q$.

The choice of locality $\mathcal{X}$ will depend on application:

- If it is expected that the affine (or quadratic) model will apply accurately to the whole image, then $\mathcal{X}$ can be the whole image and maximum robustness will be achieved.
- If not, then $\mathcal{X}$ should be a smaller region, chosen to optimise the tradeoff between robustness and model accuracy. A good way to produce a smooth field is to make $\mathcal{X}$ fairly small (e.g. a $32 \times 32$ pel region) and then to apply a smoothing filter across all the $\tilde{Q}_\mathcal{X}$ matrices, element by element, before solving for $a_{\mathcal{X},\min}$ in each region.
**Constraint Weighting Factors**

Returning to the equation for the constraint vectors, \( \mathbf{c}_i = C_i \left[ \nabla_x \theta(x_i) \begin{bmatrix} \theta_B(x_i) - \theta_A(x_i) \end{bmatrix} \right], \)

the constant gain parameter \( C_i \) will determine how much weight is given to each constraint in \( \tilde{Q}_x = \sum_{i \in \mathcal{X}} (\tilde{K}_i^T \mathbf{c}_i \mathbf{c}_i^T \tilde{K}_i) \).

Hemmendorf proposes some quite complicated heuristics for computing \( C_i \), but for the DT CWT, we find the following works well:

\[
C_i = \frac{|d_{AB}|^2}{\sum_{k=1}^{4} |u_k|^3 + |v_k|^3} \quad \text{where} \quad d_{AB} = \sum_{k=1}^{4} u_k^* v_k
\]

and \( \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix} \) and \( \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix} \) are \( 2 \times 2 \) blocks of wavelet coefficients centred on \( x_i \) in images \( A \) and \( B \) respectively.
Complex Wavelets – 39
Nick Kingsbury

Image $A$

DT CWT

Select CWT levels according to iteration

Shift within subbands

Generate displacement field $\mathbf{v}(\mathbf{x}_i)$

parameter field $\mathbf{a}_x$

Delay

Inverse DT CWT

Image $A$ registered to image $B$

Image $B$

DT CWT

Form constraints $\mathbf{c}_i$

Calculate $Q_x$ at each locality $x$

Smooth elements of $Q_x$ across image

Solve for $\mathbf{a}_{x,\text{min}}$ at each $x$

increment of parameter field $\mathbf{a}_x$
Demonstration of Registration and Image Fusion

- House on a hillside, viewed on a video camera with telephoto lens through air with significant heat turbulence (due to a hot runway).

- **Aim:** to recover the best still image from the jittery video sequence of 75 frames.

- Video sequence is courtesy of Don Fraser, Australian Defence Forces Academy, Canberra.

- **Fusion:** based on max of each wavelet coefficient magnitude across the 75 frames, combined with the mean of each coefficient’s phase.
Multi-scale Keypoint Detection using Accumulated Maps

Subject of work by Julien Fauqueur.
Rotation-Invariant Local Feature Matching

Aims:

- To derive a **local feature descriptor** for the region around a detected keypoint, so that keypoints from similar objects may be **matched reliably**.

- Matching must be performed in a **rotationally invariant** way if all rotations of an object are to be matched correctly.

- The feature descriptor must have **sufficient complexity** to give good detection reliability and low false-alarm rates.

- The feature descriptor must be **simple enough** to allow rapid pairwise comparisons of keypoints.

- Raw DTCWT coefficients provide multi-resolution local feature descriptors, but they are tied closely to a **rectangular sampling** system (as are most other multi-resolution decompositions).

Hence we first need **better rotational symmetry** for the DTCWT.
Frequency Responses of 2-D Q-shift filters at levels 3 and 4

Contours shown at $-1$ dB and $-3$ dB.
Modification of 45° and 135° subband responses for improved rotational symmetry (shown at level 4).

(a) Dual–Tree Complex Wavelets: Real Part

(b) Modified Complex Wavelets: Real Part

(c) Frequency responses of original and modified 1-D filters

(a) Original 2-D impulse responses;
(b) 2-D responses, modified to have lower centre frequencies (reduced by $1/\sqrt{1.8}$) in the 45° and 135° subbands, and even / odd symmetric real / imaginary parts;
(c) Original and modified 1-D filters.

Better rotational symmetry is achieved, but we have lost Perfect Reconstruction.
13-point circular pattern for sampling DTCWT coefs at each keypoint location

M is a precise keypoint location, obtained from the keypoint detector.

Bandpass interpolation calculates the required samples and can be performed on each subband independently because of the shift-invariance of the transform:

1. Shift by \(-\omega_1, -\omega_2\) down to zero frequency (i.e. multiply by \(e^{-j(\omega_1 x_1 + \omega_2 x_2)}\) at each point \(\{x_1, x_2\}\));

2. Lowpass interpolate to each new point (spline / bi-cubic / bi-linear);

3. Shift up by \(\{\omega_1, \omega_2\}\) (multiply by \(e^{j(\omega_1 y_1 + \omega_2 y_2)}\) at each new point \(\{y_1, y_2\}\)).
Form the Polar Matching Matrix $P$

$$P = \begin{bmatrix}
    m_1 & j_1 & k_1 & l_1 & a_1 & b_1 & c_1 \\
    m_2 & i_2 & j_2 & k_2 & l_2 & a_2 & b_2 \\
    m_3 & h_3 & i_3 & j_3 & k_3 & l_3 & a_3 \\
    m_4 & g_4 & h_4 & i_4 & j_4 & k_4 & l_4 \\
    m_5 & f_5 & g_5 & h_5 & i_5 & j_5 & k_5 \\
    m_6 & e_6 & f_6 & g_6 & h_6 & i_6 & j_6 \\
    m_6^* & d_6^* & e_6^* & f_6^* & g_6^* & h_6^* & i_6^* \\
    m_5^* & c_5^* & d_5^* & e_5^* & f_5^* & g_5^* \\
    m_4^* & b_4^* & c_4^* & d_4^* & e_4^* & f_4^* \\
    m_5^* & l_5^* & a_5^* & b_5^* & c_5^* & d_5^* & e_5^* \\
    m_6^* & k_6^* & l_6^* & a_6^* & b_6^* & c_6^* & d_6^*
\end{bmatrix}$$

Each column of $P$ comprises a set of rotationally symmetric samples from the 6 subbands and their conjugates ($^*$), whose orientations are shown by the arrows. Numbers for each arrow give the row indices in $P$. 
**Efficient Fourier-based Matching**

Columns of $P$ **shift cyclically with rotation** of the object about keypoint M. Hence we perform correlation matching in the **Fourier** domain, as follows:

- First, take 12-point FFT of each column of $P_k$ at every keypoint $k$ to give $P_k$.
- Then, for each pair of keypoints $(k, l)$ to be matched:
  - **Multiply** $P_k$ by $P_l^*$ element-by-element to give $\mathbf{S}_{k,l}$.
  - **Accumulate** the 12-point columns of $\mathbf{S}_{k,l}$ into a 48-element spectrum vector $\overline{s}_{k,l}$ (to give a 4-fold extended frequency range and hence finer correlation steps). Different columns of $\mathbf{S}_{k,l}$ are bandpass signals with differing centre frequencies, so optimum interpolation occurs if zero-padding is introduced over the part of the spectrum which is likely to contain least energy in each case.
  - Take the real part of the **inverse FFT** of $\overline{s}_{k,l}$ to obtain the 48-point correlation result $s_{k,l}$.
  - The **peak** in $s_{k,l}$ gives the **rotation and value** of the best match.
- Extra columns can be added to $P$ for multiple scales.
Correlation plots for two simple images

Each set of curves shows the output of the normalised correlator for 48 angles in $7.5^\circ$ increments, when the test image is rotated in $5^\circ$ increments from $0^\circ$ to $90^\circ$.

Levels 4 and 5 of the DTCWT were used in an 8-column $P$ matrix format.

The diameter of the 13-point sampling pattern is half the width of the subimages shown.
CORRELATION PLOTS FOR MORE COMPLICATED IMAGES
IMPROVING RESILIENCE TO ERRORS IN KEYPOINT LOCATION AND SCALE

The basic $P$-matrix normalised correlation measure is **highly resilient to changes in illumination, contrast and rotation**.

**BUT** it is still rather sensitive to discrepancies in **keypoint location and estimated dominant scale**.

To correct for small errors (typically a few pixels) in keypoint location, we modify the algorithm as follows:

- **Measure derivatives** of $\bar{P}_k$ with respect to shifts $x$ in the sampling circle.
- Using the derivatives, calculate the shift vectors $x_i$ which maximise the normalised correlation measures $s_{k,i}$ at each of the 48 rotations $i$ (using LMS methods with approximate adjustments for normalised vectors).
- By regarding the 48-point IFFT as a sparse matrix multiplication, the computation load is only **3 times** that of the basic algorithm.

We propose to do the same for small scale errors using a derivative of $\bar{P}_k$ wrt scale.
Conclusions

The Dual-Tree Complex Wavelet Transform provides shift invariance and orientation selectivity, in addition to the usual properties of the DWT. We have shown how to apply the DTCWT in the following areas:

- **Denoising** of images and 3D data to achieve performance that equals or exceeds other approaches requiring much more computation.

- **Image Registration** with an efficient multi-resolution iterative algorithm - particularly suited to non-rigid motion.

- **Rotation-invariant local feature matching** at detected keypoints for object detection and keypoint matching applications.

Papers on complex wavelets are available at:  
http://www.eng.cam.ac.uk/~ngk/

A Matlab DTCWT toolbox is available on request from: ngk@eng.cam.ac.uk