Orthonormal Hilbert-Pair of Wavelets With (Almost) Maximum Vanishing Moments

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***Abstract*—An orthonormal Hilbert-pair consists of a pair of conjugate-quadrature-filter (CQF) banks such that the equivalent wavelet function of both banks are approximate Hilbert transforms of each other. We found that the celebrated orthonormal wavelets of Daubechies, which have maximum vanishing-moment (VM), cannot be used to construct good Hilbert-pairs. In this letter, we reduce the number of VM by one and construct a Hilbert-pair with almost maximum VM. Each pair of wavelets are time-reverse versions of each other, and the individual wavelets are of the least asymmetric type (i.e., approximate linear phase CQF).**

***Index Terms*—Bernstein polynomial, complex wavelet, Hilbert-pair, orthonormal filter banks.**

I. Introduction and Preliminaries

Hilbert-pairs are becoming an important class of overcomplete transform for various signal processing applications, most notably in denoising [1]. It has roots in the seminal work of Kingsbury on dual-tree complex wavelets [2] and was later formalized by Selesnick [3], [4]. One of the main advantages of the Hilbert-pair over traditional critically sampled wavelet transform is the approximate shift-invariance in the former.

Orthonormal wavelets are obtained from a conjugate-quadrature-filter (CQF) bank. A CQF is a two-channel multirate filter bank, where the filters, denoted by $H_0(z)$ (low-pass analysis), $H_1(z)$ (high-pass analysis), $F_0(z)$ (low-pass synthesis), and $F_1(z)$ (high-pass synthesis), are obtained from a CQF filter $H(z)$ as follows: $H_0(z) = H(z)$, $H_1(z) = z^{-1}H(-z^{-1})$, $F_0(z) = H(z^{-1})$, and $F_1(z) = zH(-z)$. The CQF $H(z)$ is typically obtained from a spectral factorization of a product filter $P(z)$, i.e., $H(z)H(z^{-1}) = P(z)$. The product filter must satisfy the halfband condition $P(z) + P(-z) = 1$, and its frequency response must be nonnegative: $P(e^{j\omega}) \geq 0$. The orthonormal wavelet $\psi(t)$ (spectrum $\Psi(\omega)$) is generated from the filter bank and is given by the infinite product formula: $\Psi(\omega) = (1/2)H_1(e^{j\omega/2})\prod_{k=1}^{\infty}\{(1/2)H(e^{j\omega/2^{k+1}})\}$. To ensure convergence to a smooth function $\psi(t)$, zeros at $z = -1$ are imposed on $H(z)$, and this is also known as the vanishing moment (VM) condition [5]. VM also has roles in determining the approximation properties of the corresponding scaling function [6] and in rendering certain nonstationary random process stationary [7].

An orthonormal Hilbert-pair consists of two CQFs, denoted by $H^h(z)$ and $H^g(z)$, whose corresponding wavelets functions, denoted by $\psi^h(t)$ and $\psi^g(t)$, respectively, are Hilbert transforms of each other, i.e.,

$$\Psi^g(\omega) = \begin{cases} -j\Psi^h(\omega), & \text{for } \omega > 0 \\ j\Psi^h(\omega), & \text{for } \omega < 0 \end{cases} \tag{1}$$

where $\Psi^h(\omega)$ and $\Psi^g(\omega)$ are the Fourier transforms of $\psi^h(t)$ and $\psi^g(t)$, respectively. A sufficient condition for (1) was established by [3] to be

$$H^g(e^{j\omega}) = e^{-j\omega/2}H^h(e^{j\omega}) \tag{2}$$

and was later shown in [8] also to be necessary.

There has been several design techniques for Hilbert-pairs (both orthogonal and biorthogonal) proposed in the literature, and some (which are more relevant to this letter) will be overviewed in the next section. In this letter, we construct a class of Hilbert-pairs with one less than the maximum possible number of VM, which allows some freedom to optimize the desired Hilbert relationship. As in [9], the pair of wavelets are mirror images of each other, i.e., $\psi^g(t) = \psi^h(T-t)$ (where T is a constant), and the complex version $\psi^A(t) \equiv \psi^h(t) + j\psi^g(t)$ has the following symmetry: $\psi^A(t) = (-j\psi^A(-t))^*$.

II. Self-Hilbertian Filters

The classical approach to designing a CQF is via the spectral factorization (SF) of a product filter $P(z)$, as described earlier. Since the SF process does not yield an unique solution, there can be a variety of CQFs $H^1(z), H^2(z), \ldots$ with the same frequency response magnitude, i.e., $|H^1(e^{j\omega})| = |H^2(e^{j\omega})| = \ldots$. The phase responses are, however, different and will depend on the distribution of the zeros of $P(z)$. The extremes are the minimum phase and maximum phase cases, but the approximate linear phase case is probably the most popular. The number of possible distinct solutions increases with the length of the product filter. Any two distinct factors from the same product filter, $H^l(z)$ and $H^m(z)$, will satisfy exactly the magnitude part of (2) but will not satisfy the phase part of (2). With realizable filters, (2) can only be approximated, and this is what is desired from the two spectral factors. This motivates us to the following definition.

Definition 1: A **self-Hilbertian (SH) filter** is a product filter $P(z)$ with the following property: there exist at least two

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spectral factors, denoted by $H^l(z)$ and $H^m(z)$, whose corresponding wavelet spectrums, denoted by $\Psi^l(\omega)$ and $\Psi^h(\omega)$, satisfy

$$\Psi^l(\omega) + j\Psi^h(\omega) \approx 0, \quad \text{for} \quad \omega < 0 \tag{3}$$

i.e., the complex wavelet is approximately analytic.

An approximate orthogonal Hilbert-pair can thus be obtained from the appropriate spectral factors of an SH filter.

Definition 2: A **symmetric SH filter** has the following additional property: there exist at least two spectral factors that satisfy (3) and

$$H^l(z) = H^m(z^{-1}).$$

With the symmetric SH filter, the wavelets in the pair are mirror images of each other.

The constructions in [3] are strictly not SH, as the product filter of the corresponding CQFs are not exactly the same. The construction in [4] that employs the flat-delay all-pass filter is, however, SH. The Q-shift filters in [9] and [10] are symmetric SH. Furthermore, the individual CQF in the Q-shift solution is also approximately linear phase. Note, however, that the design techniques above do not explicitly spectrally factorize an SH filter (some spectral factorization is performed in [4], but it is not on the entire product filter).

The Hilbert-pairs to be presented later are of the symmetric SH type with approximately linear phase for the individual CQF and therefore can be called Q-shift filters. However, the filters in this letter have almost maximum VM, whereas the filters in [9] and [10] typically have a simple VM. The design technique for the filters here, which employs explicit SF of the product filter, is also very different from the techniques in [9] and [10].

We initially explored the possibility of the SH property in the maximum VM product filter (employed in the celebrated construction of Daubechies). The motivation for this is the availability of multiple spectral factors from one product filter. The number of distinct factors increases with the length of the product filter, and it was hoped that there would be at least two factors satisfying (3). Several product filters with different length (e.g., 23, 27, and 31) were tried, but unfortunately the result is negative. We therefore conclude that the maximum VM product filters cannot be SH.

The next step in our research is similar in spirit to the earlier work in [11] on biorthogonal wavelet filters. The observation is that the maximum VM condition on the product filter is too much of a constraint. We therefore reduce the number of VM by one, thus allowing one degree of freedom. This approach is detailed in the next section.

III. Product Filters via Bernstein Polynomial

A simple way to construct product filters is via the Bernstein polynomial, as the halfband and VM conditions can be structurally imposed. This approach was pioneered by Caglar and Akansu in [12] who introduced the parametric Bernstein polynomial (PBP) that is given by

$$B_N(x;\boldsymbol{\alpha}) \equiv \sum_{i=0}^{N} f(i)\binom{N}{i} x^i(1-x)^{N-i} \tag{4}$$

TABLE I
SYMMETRIC SH FILTERS WITH ALMOST MAXIMUM VM. L_{CQF}: RESULTING CQF LENGTH. NVM: NUMBER OF VM. a_1: OPTIMAL a VALUE USING E_1 MEASURE. a_2: OPTIMAL a VALUE USING E_2 MEASURE

N	L_{CQF}	nVM	a_1	a_2	$E_{1,min}(\%)$	$E_{2,min}(\%)$
3	4	1	0.2589	0.2631	13.23	3.95
5	6	2	0.2965	0.2913	10.61	1.31
7	8	3	0.0460	0.0460	3.06	0.088
9	10	4	0.4231	0.4040	11.64	1.26
11	12	5	0.1824	0.1827	4.07	0.17
13	14	6	0.0487	0.0511	10.21	1.15
15	16	7	0.2588	0.2587	5.58	0.33
17	18	8	0.2407	0.2405	4.96	0.15
19	20	9	0.1307	0.1339	7.86	0.52
21	22	10	0.0240	0.0245	5.40	0.22

where N is odd, $\boldsymbol{\alpha} = [\alpha_0 \ \dots \ \alpha_{(N-1)/2}]^T$ are the Bernstein parameters, and

$$f(i) \equiv \begin{cases} 1-\alpha_i & 0 \le i \le \frac{1}{2}(N-1) \\ \alpha_{N-i} & \frac{1}{2}(N+1) \le i \le N. \end{cases} \tag{5}$$

The polynomial can be transformed into a z-transform filter function by the following substitution: $x = -(1/4)z(1-z^{-1})^2 = \sin^2(\omega/2)$ and satisfies the halfband filter condition: $B(x) + B(1-x) = 1$, where for brevity $B(x) = B_N(x;\boldsymbol{\alpha})$. If $\alpha_i = 0$ for $i = 0, \dots, L$, then $B(x)$ has $(L+1)$ zeros at $x = 1$, i.e., $B(x) = (1-x)^{L+1}R(x)$, where $R(x)$ is the remainder polynomial. The resulting $P(z) = B(-(1/4)z(1-z^{-1})^2)$, which is of length $2N+1$, will have $2(L+1)$ zeros at $z = -1$. This means that the desired number of VM can be easily imposed, and this is one of the main appeals of the PBP. With $L = L_{max} \equiv (N-1)/2$, the PBP gives the maximum VM product filters of Daubechies.

In this letter, we set all Bernstein parameters to zero except one, i.e.,

$$\alpha_0 = \alpha_1 = \ldots = \alpha_{(N-1)/2-1} = 0.$$

The resulting CQF $H(z)$ will be of length $N+1$ with $(N-1)/2$ VMs. For brevity, we denote the nonzero parameter by $a \equiv \alpha_{(N-1)/2}$ and restrict its values to $0 \le a \le 0.5$ to ensure $B(x) \ge 0$ for $0 \le x \le 1$. This restriction is similar to that imposed in [12] to the more general case of multiple nonzero parameters, but a recent work [13] showed that, in general, $0 \le \alpha_i \le 1/2$ is not necessary and is in fact too restrictive. However, we are dealing with the special case of only one parameter so this restriction will be maintained to ensure nonnegativity in the frequency response.

We thus have a one parameter family of product filters with almost maximum VM. The task is then to determine a suitable value of a to give a symmetric SH filter. Since there is only one parameter a with limited range $a \in [0, 0.5]$, the entire range can be examined (scanned) to determine the optimal value. Two measures of optimality will be considered here. By denoting

TABLE II
COEFFICIENTS OF CQF FOR HILBERT-PAIRS. L_{CQF}: CQF LENGTH. NVM: NUMBER OF VM. THE COEFFICIENTS ARE NORMALIZED TO UNITY DC GAIN

L_{CQF}	nVM	Filter coefficients
8	3	-0.0621, 0.0039, 0.4132, 0.5466, 0.1479, -0.0670, 0.0010, 0.0165
12	5	-0.0054, 0.0101, 0.0325, -0.0455, -0.0087, 0.3753, 0.5530, 0.2038, -0.0783, -0.0474, 0.0069, 0.0037
18	8	-0.0003, -0.0023, 0.0032, 0.0221, -0.0128, -0.1020, 0.0290, 0.4217, 0.5243, 0.1698, -0.0706, -0.0065, 0.0364, -0.0041, -0.0105, 0.0015, 0.0013, -0.0002
22	10	0.0003, 0.0006, -0.0034, -0.0047, 0.0187, 0.0273, -0.0546, -0.0733, 0.2144, 0.5320, 0.3908, 0.0111 -0.0848, 0.0197, 0.0256, -0.0178, -0.0087, 0.0062, 0.0018, -0.0013, -0.0002, 0.0001

$\Psi^A(\omega) \equiv \Psi^l(\omega) + j\Psi^h(\omega)$, the first is a L^∞ measure defined as

$$E_1 \equiv \frac{\max_{\omega<0} \left|\Psi^A(\omega)\right|}{\max_{\omega>0} \left|\Psi^A(\omega)\right|}$$

which measures the peak error. The second is a L^2 measure defined as

$$E_2 \equiv \frac{\int_{-\infty}^{0} \left|\Psi^A(\omega)\right|^2 d\omega}{\int_{0}^{\infty} \left|\Psi^A(\omega)\right|^2 d\omega}$$

which measures the negative frequency energy. The construction procedure can be summarized as follows.

1) Choose a sufficiently fine grid for the discretization of a over the range [0,0.5].
2) For each value of a, determine the product filter $P(z)$.
3) Spectral factorize $P(z)$ using a roots distribution strategy that will give an approximately linear phase CQF $H(z)$. The strategy alternately chooses roots inside and outside the unit circle [5].
4) Compute the equivalent wavelet functions corresponding to the CQF $H(z)$ and the time reverse CQF $H(z^{-1})$ using the classical tree-structured iterated filter bank [5], [6].
5) Calculate the optimality measure, either E_1 or E_2, for each a and choose the CQF $H(z)$ with the smallest measure.

IV. EXAMPLES

Using the procedure described in the previous section, we computed the optimal a values for product filters of different length. The results are shown in Table I. Several comments are in order, as follows.

1) The optimal a's using either the E_1 or E_2 measure are about the same. There does not seem to be any general pattern to the optimal a value as the length increases.
2) There is also no pattern to the quality of approximation (to the Hilbert transform) as the length increases. The first two filters, of length 4 and 6, are too short to give a good approximation.
3) We can broadly classify the filters into the following three classes based on the quality of approximation:
   a) *Very good (VG)*: $N = 7, 11, 17, 21$ with length 8, 12, 18, 22, respectively.
   b) *Good (G)*: $N = 15, 19$ with length 16, 20, respectively.
   c) *Average (A)*: $N = 3, 5, 9, 13$ with length 4, 6 10, 14, respectively.

   Surprisingly, a short filter $N = 7$ (length 8) has the best Hilbert approximation, although it is not very smooth.

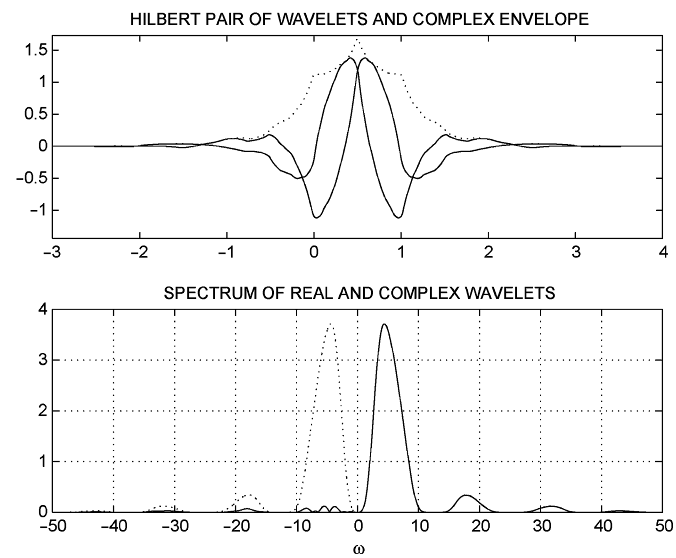

Fig. 1. Orthonormal Hilbert-pair from length 8 CQF using $a = 0.046$. Top diagram: time wavelet functions (solid line) and the magnitude of complex envelope $|\psi^h(t) + j\psi^g(t)|$ (dotted line). Bottom diagram: spectrum of complex wavelet $|\Psi^h(\omega)+j\Psi^g(\omega)|$ (solid line) and spectrum of real wavelet $2|\Psi^h(\omega)|$.

The coefficients of the CQF for the VG cases are listed in Table II. Plots of the time functions and spectrums of the wavelets corresponding to the VG cases (using a_1 values) are shown in Figs. 1–4. From the spectrums, we readily see that approximate complex analytic functions are achieved with the wavelet pairs. As expected, the smoothness of the time wavelet function increases with the number of VM.

V. CONCLUDING REMARKS

The concept of the SH filter was introduced here, and its role in the construction of orthonormal Hilbert-pair of wavelets was explained. It was found that maximum VM product filters cannot be SH. However, by reducing the number of VM by one from the maximum, one degree of freedom can be introduced. This degree of freedom can be used to tune the characteristics of the product filter to achieve the SH property. Using this approach, a class of orthonormal Hilbert-pairs with almost maximum VM was constructed. The wavelets in each pair are mirror

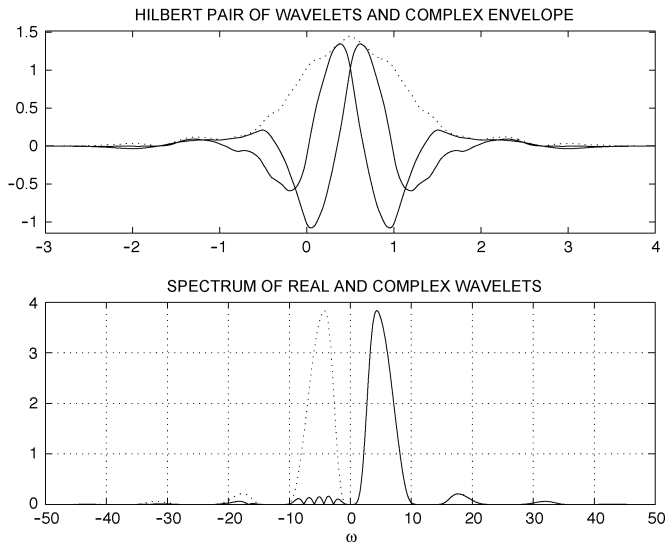

Fig. 2. Orthonormal Hilbert-pair from length 12 CQF using $a = 0.1824$. Top diagram: time wavelet functions (solid line) and the magnitude of complex envelope $|\psi^h(t) + j\psi^g(t)|$ (dotted line). Bottom diagram: spectrum of complex wavelet $|\Psi^h(\omega)+j\Psi^g(\omega)|$ (solid line) and spectrum of real wavelet $2|\Psi^h(\omega)|$.

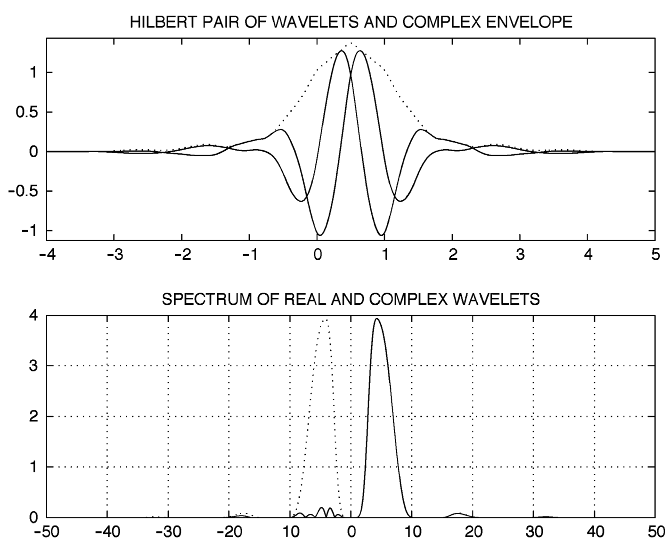

Fig. 3. Orthonormal Hilbert-pair from length 18 CQF using $a = 0.2407$. Top diagram: time wavelet functions (solid line) and the magnitude of complex envelope $|\psi^h(t) + j\psi^g(t)|$ (dotted line). Bottom diagram: spectrum of complex wavelet $|\Psi^h(\omega)+j\Psi^g(\omega)|$ (solid line) and spectrum of real wavelet $2|\Psi^h(\omega)|$.

images of each other, and the individual wavelet is of the least asymmetric type.

Further work in this direction includes allowing more degrees of freedom by further reducing the number of VM. Can this yield a better approximation to the Hilbert transform? If yes, what is the strategy that should be used to determine the values of the nonzero Bernstein parameters?

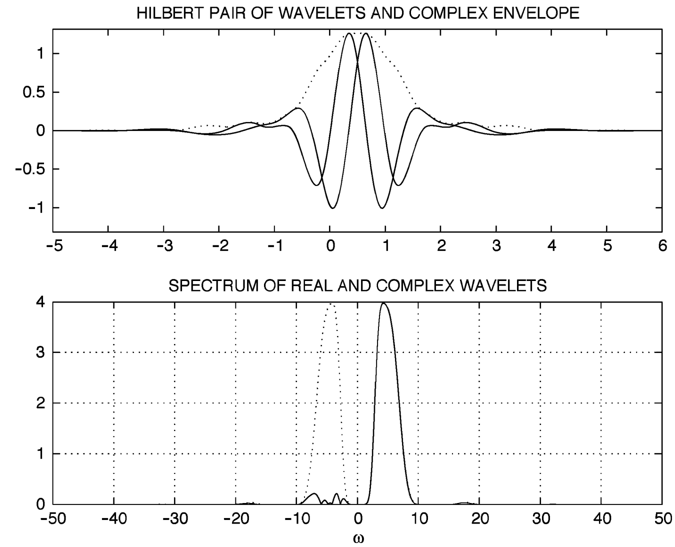

Fig. 4. Orthonormal Hilbert-pair from length 22 CQF using $a = 0.0240$. Top diagram: time wavelet functions (solid line) and the magnitude of complex envelope $|\psi^h(t) + j\psi^g(t)|$ (dotted line). Bottom diagram: spectrum of complex wavelet $|\Psi^h(\omega)+j\Psi^g(\omega)|$ (solid line) and spectrum of real wavelet $2|\Psi^h(\omega)|$.

References

[1] I. W. Selesnick, R. G. Baraniuk, and N. G. Kingsbury, "The dual-tree complex wavelet transform," *IEEE Signal Process. Mag.*, vol. 22, no. 6, pp. 123–151, Nov. 2005.

[2] N. Kingsbury, "Complex wavelets for shift invariant analysis and filtering of signals," *J. Appl. Comput. Harmon. Anal.*, vol. 10, no. 3, pp. 234–253, May 2001.

[3] I. W. Selesnick, "Hilbert transform pairs of wavelet bases," *IEEE Signal Process. Lett.*, vol. 8, no. 6, pp. 170–173, Jun. 2001.

[4] ——, "The design of approximate Hilbert transform pairs of wavelet bases," *IEEE Trans. Signal Process.*, vol. 50, no. 5, pp. 1144–1152, May 2002.

[5] I. Daubechies, *Ten Lectures on Wavelets*. Philadelphia, PA: SIAM, 1992.

[6] G. Strang and T. Nguyen, *Wavelets and Filter Banks*. Cambridge, MA: Wellesley-Cambridge Press, 1996.

[7] P. F. Craigmile and D. B. Percival, "Asymptotic correlation of between-scale wavelet coefficients," *IEEE Trans. Inf. Theory*, vol. 51, no. 3, pp. 1039–1048, Mar. 2005.

[8] H. Ozkaramanli and R. Yu, "On the phase condition and its solution for Hilbert transform pairs of wavelet bases," in *Proc. ICASSP*, 2003.

[9] N. Kingsbury, "A dual-tree complex wavelet transform with improved orthogonality and symmetry properties," in *Proc. IEEE Int. Conf. Image Processing (ICIP)*, 2000.

[10] ——, "Design of Q-shift complex wavelets for image processing using frequency domain energy minimization," in *Proc. IEEE Int. Conf. Image Processing (ICIP)*, 2003.

[11] D. B. H. Tay, "Rationalizing the coefficients of popular biorthogonal wavelet filters," *IEEE Trans. Circuits Syst. Video Technol.*, vol. 10, no. 6, pp. 998–1005, Sep. 2000.

[12] H. Caglar and A. N. Akansu, "A generalized parametric PR-QMF design technique based on Bernstein polynomial approximation," *IEEE Trans. Signal Process.*, vol. 41, no. 7, pp. 2314–2321, Jul. 1993.

[13] D. B. H. Tay, "Zero-pinning the Bernstein polynomial: a simple design technique for orthonormal wavelets," *IEEE Signal Process. Lett.*, vol. 12, no. 12, pp. 835–838, Dec. 2005.