# Final Assignment for 5R1, Stochastic Processes.

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## 1 Gaussian Model

In this exercise you will implement a Gibbs sampler for a toy model described in the lecture.

$$\begin{aligned} s_1 &\sim p(s_1) = \mathcal{N}(s_1; \mu_1, P_1) \\ s_2 &\sim p(s_2) = \mathcal{N}(s_2; \mu_2, P_2) \\ x|s_1, s_2 &\sim p(x|s_1, s_2) = \mathcal{N}(x; s_1 + s_2, R) \end{aligned}$$

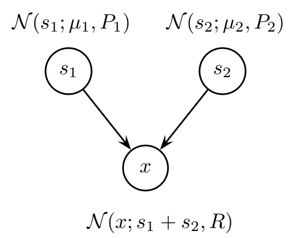



Figure 1: OSSS model

We will use the following parameters: $\mu_1 = 3$, $\mu_2 = 5$, $P_1, P_2 = 0.5$ and $R = 0.3$. The Gaussian with mean $m$ and variance $r$ is denoted as

$$\mathcal{N}(x; m, r) = \exp\{-\frac{1}{2}(x^2 + m^2 - 2xm)/r - \frac{1}{2}\log(2\pi r)\}$$

### 1.1 Forward sampling

1. Write a matlab or octave program to sample from the model.
2. Find $p(x)$ analytically and verify your result by plotting an histogram of the samples generated.

Submit your program listing and the figure. Hint: In Matlab, you can sample from a univariate Gaussian $\mathcal{N}(x; m, r)$ by $\texttt{sqrt(r)}.*\texttt{randn}+\texttt{m}$ where `randn` is the unit normal random number generator with $\mathcal{N}(z; 0, 1)$.

## 1.2 Exact result

Suppose we observe $x = 9$.

1. Find $p(s_1, s_2|x = 9)$ analytically. Submit your derivation.

$$
\begin{aligned}
\tilde{m} &= \begin{pmatrix} 3.3846 \\ 5.3846 \end{pmatrix} \\
\Sigma &= \begin{pmatrix} 0.3077 & -0.1923 \\ -0.1923 & 0.3077 \end{pmatrix}
\end{aligned}
$$

Hint: Follow the derivations in the lecture slides. The posterior is Gaussian with mean $\tilde{m}$ and covariance $\Sigma$.

## 1.3 Gibbs sampling

1. Find the full conditionals analytically $p(s_1|s_2^{(t)}, x = \hat{x})$ and $p(s_2|s_1^{(t)}, x = \hat{x})$

2. Implement a Gibbs sampler

$$
\begin{aligned}
s_1^{(t)} &\sim p(s_1|s_2^{(t)}, x = \hat{x}) \\
s_2^{(t+1)} &\sim p(s_2|s_1^{(t)}, x = \hat{x})
\end{aligned}
$$

3. Run the sampler for $t = 1 \ldots T$ steps. Compute the sample mean $\tilde{s}_T$ and the sample covariance matrix $\tilde{\Sigma}_T$ of the generated samples $s^{(t)} = (s_1^{(t)}, s_2^{(t)})^\top$.

$$
\begin{aligned}
\tilde{s}_T &= \frac{1}{T}\sum_{t=1}^{T} s^{(t)} \\
\tilde{\Sigma}_T &= \frac{1}{T}\sum_{t=1}^{T} (s^{(t)} - \tilde{s}_T)(s^{(t)} - \tilde{s}_T)^\top
\end{aligned}
$$

Plot the entries of the sample mean and the sample covariance matrix as a function of $T$. Observe if the estimates converge to the exact posterior mean and covariance for increasing $T$?

Submit your derivation, program listing and figures.

Hint: One full conditional is given as

$$
\begin{aligned}
p(s_1|s_2^{(t)}, x = \hat{x}) &= \mathcal{N}(s_1; \tilde{m}_{1|2}, \Sigma_{1|2}) \\
\Sigma_{1|2}^{-1} &= P_1^{-1} + R^{-1} \\
\tilde{m}_{1|2} &= \Sigma_{1|2}\left(P_1^{-1}\mu_1 + R^{-1}(\hat{x} - s_2^{(t)})\right)
\end{aligned}
$$

## 1.4 Slow Convergence when two variables are strongly dependent

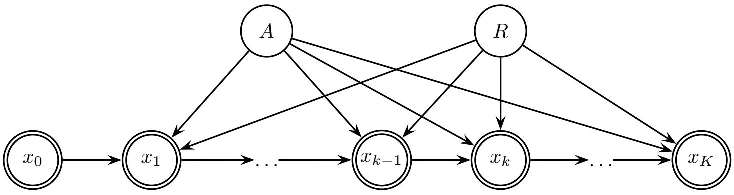
Repeat the previous experiment with $R = 0.005$. How many iterations does it take until convergence if $\theta^{(0)} = (\mu_1, \mu_2)$?

Hint: Exact numerical result is

$$\begin{aligned}
\tilde{m} &= \begin{pmatrix} 3.4975 \\ 5.4975 \end{pmatrix} \\
\Sigma &= \begin{pmatrix} 0.2512 & -0.2488 \\ -0.2488 & 0.2512 \end{pmatrix}
\end{aligned}$$

## 2 AR(1) Model

In this problem, you will develop a Gibbs sampler for the AR model:




$$
\begin{aligned}
A &\sim \mathcal{N}(A; 0, 1.2) \\
R &\sim \mathcal{IG}(R; 0.4, 250) \\
x_k | x_{k-1}, A, R &\sim \mathcal{N}(x_k; A x_{k-1}, R) \\
x_0 &= 1 \qquad\qquad x_1 = -6
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{N}(x; m, r) &= \exp\{-\frac{1}{2}(x^2 + m^2 - 2xm)/r - \frac{1}{2}\log(2\pi r)\} \\
\mathcal{IG}(r; a, b) &= \exp\left(-(a+1)\log r - \frac{1}{br} - \log\Gamma(a) - a\log b\right)
\end{aligned}
$$

1. Derive the full conditional distributions $p(A|R, x_0, x_1)$ and $p(R|A, x_0, x_1)$
2. Implement the Gibbs sampler. Plot the generated samples $(A^{(t)}, R^{(t)})$ for $t = 1 \dots T$ as a scatterplot.
3. Estimate the expected values of $\langle A|x_0, x_1\rangle$ and $\langle R|x_0, x_1\rangle$ from the generated samples

Submit your listing, the figure, the estimates and the derivations.

Hint: To generate an inverse gamma random variable, you can use `1/(b * gamrnd(a, 1))` or `1/(gamrnd(a, b))`, where `gamrnd`$(a, b)$ is a function that samples a gamma random variable with shape $a$ and scale $b$.

You are expected to submit at least the implementations and simulation results. Extra credit will be given for clear and detailed derivations.