The Entropic Central Limit Theorem for Discrete Random Variables

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Outline

1. Information Theoretic CLTs
   - The Entropic CLT
2. Discrete Random Variables
3. On Monotonicity
4. Proof
   - Bernoulli Smoothing

Information Theoretic CLTs

Let $X_1, \ldots, X_n$ be i.i.d. random variables with mean 0 and finite variance σ^2 and let $\hat{S}_n = \frac{1}{\sqrt{n}} \sum_{i=1}^n X_i$ denote their standardised sum.

Information theory and the central limit theorem (CLT) have a long history starting with [Linnik '59], [Shimizu '75] and [Brown '82]

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The *Fisher information* of X with density f is $I(X) = \int f(\frac{f'}{f})^2$.
For X, Y independent

$$I(X+Y) \leq I(X) \quad \text{(convolution inequality)}$$

Information Theoretic CLTs: The Entropic CLT

The *differential entropy* of X is $h(X) = -\int f \log f$
and the relative entropy between X and Y with densities f and g is
$D(X||Y) = D(f||g) = \int f \log \frac{f}{g}$

Let Z be a 0 mean Gaussian with variance σ^2. Then

$$\begin{aligned} D(X) := D(X||Z) &= h(Z) - h(X) \\ &= \frac{1}{2}\log\left(2\pi e \sigma^2\right) - h(X) \end{aligned}$$

and by positivity of relative entropy

$$h(X) \leq h(Z) \quad \text{(Gaussian maximum entropy)}$$

Information Theoretic CLTs: The Entropic CLT

Theorem (Entropic CLT [Barron, '86])

Let $\hat{S}_n$ denote the standardised sum of i.i.d. $X_1, \ldots, X_n$

$$D(\hat{S}_n) \to 0 \quad \text{if and only if } D(\hat{S}_n) \text{ is finite for some } n$$

Equivalently, writing $S_n = \sum_{i=1}^n X_i$,

$$h(\hat{S}_n) = h(S_n) - \log \sqrt{n} \to \frac{1}{2} \log (2\pi e \sigma^2)$$

By Pinsker's inequality

$$\|\hat{S}_n - Z\|_{\mathrm{TV}} \to 0$$

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What about discrete?

Discrete Random Variables

The *entropy* of a discrete random variable Y with PMF p on A is $H(Y) = -\sum_{y \in A} p(y) \log p(y)$

Y has a lattice distribution with span $h > 0$ if its support is a subset of $\{a + kh \ : \ k \in \mathbb{Z}\}$ for some $a \in \mathbb{R}$
h is *maximal* if it is the largest such h.

Let $\{X_n\}$ be i.i.d. lattice with variance σ^2 and maximal span h.
Let $S_n = \sum_{i=1}^{n} X_i$
Unlike the differential entropy

$$H\Big(\frac{1}{\sqrt{n}} S_n\Big) = H(S_n) \to \infty$$

Discrete Random Variables

Theorem (Entropy convergence)

$$\lim_{n\to\infty}\left[H(S_n) - \log\frac{\sqrt{n}}{h}\right] = \frac{1}{2}\log(2\pi e\sigma^2)$$

Discrete Random Variables

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But using strong forms of the CLT.

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In fact, this theorem implies the CLT!

Discrete Random Variables

Suppose Y is lattice with PMF p, maximal span h,
values in $A = \{a + kh \ : \ k \in \mathbb{Z}\}$, mean μ, and variance σ^2

Definition (Discrete Gaussianity)

Define

$$D(Y) := D(p\|q) = \sum_{k\in\mathbb{Z}} p(a+kh) \log \frac{p(a+kh)}{q(a+kh)}$$

where q is the PMF of a Gaussian $Z \sim N(\mu, \sigma^2)$ quantised on A as,

$$q(a+kh) = \int_{a+kh}^{a+(k+1)h} \phi(x)dx, \qquad k \in \mathbb{Z},$$

where ϕ is the $N(\mu, \sigma^2)$ density.

By definition, $D(Y + c) = D(Y)$ for any constant c.

Discrete Random Variables

Theorem (Discrete entropic CLT)

If $X_1, X_2, \ldots$ are i.i.d. lattice and $\hat{S}_n = \frac{1}{\sqrt{n}} \sum_{i=1}^n X_i$, then

$$D(\hat{S}_n) \to 0, \qquad \text{as } n \to \infty.$$

Take WLOG $\mu = 0$, let $Z \sim N(0, \sigma^2)$ and let Z_n be the quantised Gaussian. Then, from Pinsker's and the triangle inequality for the total variation norm

$$\|\hat{S}_n - Z\|_{\mathrm{TV}} \leq \sqrt{\frac{1}{2} D(\hat{S}_n)} + \|Z_n - Z\|_{\mathrm{TV}} \to 0 \quad \text{(strong version of CLT)}$$

Alternatively, $\|\hat{S}_n - Z_n\|_{\mathrm{TV}} \to 0$

Discrete Random Variables

Theorem (Entropy-relative entropy solidarity)

$$D(\hat{S}_n) = \frac{1}{2}\log(2\pi e\sigma^2) - \left[H(S_n) - \log\frac{\sqrt{n}}{h}\right] + O\left(\frac{1}{\sqrt{n}}\right)$$

By positivity of the relative entropy,

$H(S_n) - \log\frac{\sqrt{n}}{h} \leq \frac{1}{2}\log(2\pi e\sigma^2) + O\left(\frac{1}{\sqrt{n}}\right)$,
so the standardised entropy converges to its maximum limit!

On Monotonicity

- Continuous

  Entropy Power Inequality (EPI):

$$h(X_1 + X_2) \geq h(X_1) + \frac{1}{2}\log 2$$

$\Rightarrow h(\hat{S}_{2n}) \geq h(\hat{S}_n)$ for all n

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- Discrete
  $H(X_1 + X_2) \geq H(X_1) + \frac{1}{2}\log 2$ fails in general.
  However, for i.i.d. X_1, X_2

$$H(X_1 + X_2) \geq H(X_1) + \frac{1}{2}\log 2 - o_{H(X_1)}(1), \quad \text{[Tao, '10]}$$

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2. "Bernoulli smoothing"
   If $\{V_n\}$ are i.i.d. lattice and $\{B_n\}$ i.i.d. $\mathrm{Bern}(1/2)$ independent,
   $$H\left(\sum_{i=1}^{n} [V_i + B_i] \right) - \log \sqrt{n} \to \frac{1}{2} \log \left(2\pi e \left(\sigma_V^2 + \frac{1}{4} \right) \right)$$

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3. Bernoulli part decomposition

$$S_n \stackrel{\mathcal{D}}{=} V^{(n)} + W^{(n)}B,$$

   for some lattice $V^{(n)}$, $W^{(n)} \sim \mathrm{Bern}(q^{(n)})$ with $q^{(n)} \to 1$

Proof: Bernoulli Smoothing

Lemma

Let U be an independent uniform on $(-1/2, 1/2)$. Then

$$D(\hat{S}_n) = D\Big(\hat{S}_n + \frac{1}{\sqrt{n}}U\Big) + O\Big(\frac{1}{\sqrt{n}}\Big)$$

as $n \to \infty$.

Proof: Bernoulli Smoothing

Standardised Fisher information: $J(X) := \mathrm{Var}(X)I(X) - 1$

de Bruijn's identity: $D(X) = \int_0^1 J(\sqrt{1-t}X + \sqrt{t}Z)\frac{dt}{2(1-t)}$

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$$D\left(\frac{1}{\sqrt{n}}\left[\sum_{i=1}^n V_i + B_i\right] + \frac{1}{\sqrt{n}}U\right) = D\left(\frac{1}{\sqrt{2n}}\left[\hat{S}_n + U\right] + \frac{1}{\sqrt{2}}Z\right)$$

$$+ \int_0^{1/2} J\left(\sqrt{\frac{1-t}{n}}\left[\sum_{i=1}^n V_i + B_i\right] + \sqrt{\frac{1-t}{n}}U + \sqrt{t}Z\right)\frac{dt}{2(1-t)},$$

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- The integrand vanishes for each fixed $t \in (0,1)$ by the results of [Barron, '86] and, by the convolution inequality, is $\leq \left(1 + \frac{\sigma_V^2}{\sigma'^2}\right)J\left(\sqrt{\frac{1-t}{n}}\sum_{i=1}^n B_i + \sqrt{\frac{1-t}{n}}U + \sqrt{t}Z'\right) + \frac{\sigma_V^2}{\sigma'^2}$, whose integral vanishes by the binomial case (Step 1)!

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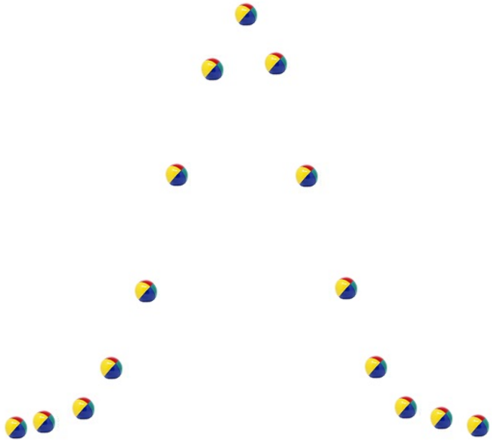
$\Rightarrow$ Uniform integrability

Further Work

- Non-lattice
- Rates of convergence under additional moment assumptions
- (Approximate) Monotonicity (of any of the quantities appearing in the proof)
- Dependent random variables
- Random vectors



Thank you!