4F8 Image Coding – Examples Solutions

1. Level 1 of the Haar transform on 8-pt vectors is:

$$T_1 = \begin{bmatrix} T & & & 0 \\ & T & & \\ & & T & \\ 0 & & & T \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & & & & & & 0 \\ 1 & -1 & & & & & & \\ & & 1 & 1 & & & & \\ & & 1 & -1 & & & & \\ & & & & 1 & 1 & & \\ 0 & & & & 1 & -1 & & \\ & & & & & & 1 & 1 \\ & & & & & & 1 & -1 \end{bmatrix}$$

Reordering so that all the lowpass filters (1 1) come first, & then applying T to these gives the 2-level transform as:

$$T_2 = \begin{bmatrix} T & & & 0 \\ & T & & \\ & & I_2 & \\ 0 & & & I_2 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & & & & & & 0 \\ & & 1 & 1 & & & & \\ & & & & 1 & 1 & & \\ & & & & & & 1 & 1 \\ 1 & -1 & & & 0 & & & \\ & & 1 & -1 & & & & \\ & & & & 1 & -1 & & \\ 0 & & & & & & 1 & -1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 & & & & \\ 1 & 1 & -1 & -1 & & & & \\ & & & & 1 & 1 & 1 & 1 \\ & & & & 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & & & & & & \\ & & \sqrt{2} & -\sqrt{2} & & & & \\ & & & & \sqrt{2} & -\sqrt{2} & & \\ & & & & & & \sqrt{2} & -\sqrt{2} \end{bmatrix}$$

Repeating this process on the 1st of 4 rows:

$$T_3 = \begin{bmatrix} T & & & \\ & I_2 & & \\ & & I_2 & \\ & & & I_2 \end{bmatrix} \cdot \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 & & & & \\ & & & & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & & & & \\ & & & & 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & & & & & & \\ & & \sqrt{2} & -\sqrt{2} & & & & \\ & & & & \sqrt{2} & -\sqrt{2} & & \\ & & & & & & \sqrt{2} & -\sqrt{2} \end{bmatrix} = \frac{1}{2\sqrt{2}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & & & & \\ & & & & \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} \\ 2 & -2 & & & & & & \\ & & 2 & -2 & & & & \\ & & & & 2 & -2 & & \\ & & & & & & 2 & -2 \end{bmatrix}$$

Orthogonality: Any row $\times$ [Any other row]$^T = 0$

eg. row $1 \times [\text{row } 3]^T = \sqrt{2} + \sqrt{2} - \sqrt{2} - \sqrt{2} = 0$

row $1 \times [\text{row } 5]^T = 2 - 2 = 0$

row $3 \times [\text{row } 5]^T = 2\sqrt{2} - 2\sqrt{2} = 0$

~~Similarly for all~~

row $5 \times [\text{row } 6]^T = 0$

row $1 \times [\text{row } 2]^T = 1 + 1 + 1 + 1 - 1 - 1 - 1 - 1 = 0$

Similarly for all other combinations.

Normalisation: Rows 1 & 2: $\sum_{i=1}^{8} \left(\frac{1}{2\sqrt{2}}\right)^2 = 1$

Rows 3 & 4: $\sum_{i=1}^{4} \left(\frac{\sqrt{2}}{2\sqrt{2}}\right)^2 = 1$

Rows 5 $\rightarrow$ 8: $\sum_{i=1}^{2} \left(\frac{2}{2\sqrt{2}}\right)^2 = 1$

$\therefore$ Matrix is orthonormal

For image compression apply the transform to the rows & columns of ~~the~~ each 8×8 pel region X ~~each region of pels~~ in the

using $$Y = T_3 X T_3^T$$

then quantise Y apply entropy coding to obtain the minimum bit rate.

To recover X for each block use:

$$X = T_3^T Y T_3.$$

~~2. N.B. There is an error in the notes and in the Examples Paper: $t_{ki} = \sqrt{\frac{2}{n}} \cos(\ldots)$ and not $\frac{2}{\sqrt{n}}$ as given.~~ Corrected for 1997 NGK.

2. Orthogonality:

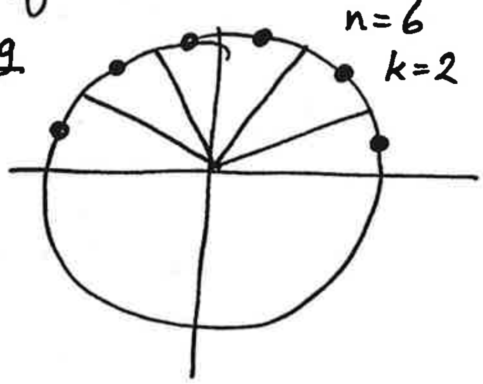
First consider $\underline{t}_1 \cdot \underline{t}_k^T = \sum_{i=1}^{n} \frac{\sqrt{2}}{n} \cos\left(\frac{\pi(2i-1)(k-1)}{2n}\right)$

for $2 \leq k \leq n$.

The angles $\frac{\pi(2i-1)(k-1)}{2n}$ correspond to the centres of n sectors, uniformly spaced around the unit circle, covering the range 0 to $(k-1)\pi$.

Hence the cos terms are proportional to the projections of the sector centres of gravity onto the real axis. From symmetry considerations these projections onto the real axis will sum to zero for all integer $(k-1)$ as long as $(k-1)$ is non-zero and not an integer multiple of $2n$.

e.g



Other proofs are possible, but are less intuitive.

Consider $t_{li} \times t_{ki}^T = \frac{2}{n} \sum_{i=1}^{n} \cos \frac{\pi(2i-1)(l-1)}{2n} \cos \frac{\pi(2i-1)(k-1)}{2n}$

$$= \frac{1}{n} \sum_{i=1}^{n} \cos \frac{\pi(2i-1)(l+k-2)}{2n} + \frac{1}{n} \sum_{i=1}^{n} \cos \frac{\pi(2i-1)(l-k)}{2n}$$

As long as $l+k-2$ is not zero & not an integer multiple of $2n$, the first Σ will be zero.

As long as $l-k$ is not zero & not an integer of $2n$, the second Σ will be zero.

But l & k only go from 2 to n & $l \neq k$ ~~are not the same~~ so all rows are orthogonal.

Normalisation:

$$\sum_{i=1}^{n} t_{1i}^2 = n \cdot \frac{1}{n} = 1 \quad \therefore \text{ row 1 is normalised}$$

$$\sum_{i=1}^{n} t_{ki}^2 = \frac{2}{n} \sum_{i=1}^{n} \cos^2\left(\frac{\pi(2i-1)(k-1)}{2n}\right)$$

$$= \frac{1}{n} \sum_{i=1}^{n} \left[1 + \cos\left(\frac{2\pi(2i-1)(k-1)}{2n}\right)\right]$$

$$= n \cdot \frac{1}{n} + 0 = 1 \quad \therefore \text{ rows } 2 \to n \text{ are normalised}$$

Orthonormality allows the transform to be easily inverted since $T^{-1} = T^T$ and signal energy ~~are~~ is preserved.

3. ~~DF~~ $2n$-pt DFT:

$$y_{k+1} = \frac{1}{\sqrt{2n}} \sum_{i=0}^{2n-1} x_{i+1} e^{-j.ki.2\pi/2n}$$

$$= \frac{1}{\sqrt{2n}} \sum_{i=0}^{n-1} \Big(x_{i+1} e^{-j.ki.2\pi/2n} \cancel{+ x_{i+1} e^{-jk(2n-1-i)}} + x_{i+1} e^{-jk(2n-1-i).2\pi/2n} \Big)$$

since $x_{i+1} = x_{2n-i}$ for $i = 0$ to $n-1$.

$$\therefore y_{k+1} = \frac{1}{\sqrt{2n}} \sum_{i=0}^{n-1} x_{i+1} \left(e^{-jki\pi/n} + e^{jk(i+1)\pi/n} \right)$$

$$= \frac{1}{\sqrt{2n}} \sum_{i=0}^{n-1} x_{i+1} . e^{jk\pi/2n} \left(e^{-jk(2i+1)\pi/2n} + e^{jk(2i+1)\pi/2n} \right)$$

$$= \sqrt{\frac{2}{n}}\, e^{jk\pi/2n} . \sum_{i=0}^{n-1} x_{i+1} \cos\left(\frac{k(2i+1)\pi}{2n}\right)$$

For the DCT:

$$\tilde{y}_{k+1} = \sqrt{\frac{2}{n}} \sum_{i=1}^{n} x_i \cos\left(\frac{\pi(2i-1)k}{2n}\right)$$

$$= \sqrt{\frac{2}{n}} \sum_{i=0}^{n-1} x_{i+1} \cos\left(\frac{k(2i+1)\pi}{2n}\right)$$

$\therefore$ DCT coef $\tilde{y}_{k+1}$ = DFT coef $y_{k+1} . e^{-jk\pi/2n}$

which is equivalent to a $\frac{1}{2}$ sample period shift in the data x.

4. For valid pdf: $\int_{-\infty}^{\infty} p(x)\,dx = 1$

$$\int_{-\infty}^{\infty} c\,e^{-|x|/x_0}\,dx = 2\int_0^{\infty} c\,e^{-x/x_0}\,dx = 2\left[-x_0 c\,e^{-x/x_0}\right]_0^{\infty}$$

$$= 2x_0 c = 1 \qquad \therefore\ c = \underline{\underline{\frac{1}{2x_0}}}$$

Probability of x between x_1 & x_2 is:

$$\int_{x_1}^{x_2} p(x)\,dx = \int_{x_1}^{x_2} c\,e^{-x/x_0}\,dx \quad \text{if } x_2 \geqslant x_1 \geqslant 0$$

$$= \left[-\tfrac{1}{2}e^{-x/x_0}\right]_{x_1}^{x_2} = \tfrac{1}{2}\left(e^{-x_1/x_0} - e^{-x_2/x_0}\right)$$

If $k > 0$, $p_k = \int_{(k-\frac{1}{2})q}^{(k+\frac{1}{2})q} p(x)\,dx =$

$$= \underline{\underline{\tfrac{1}{2}\left(e^{-(k-\frac{1}{2})q/x_0} - e^{-(k+\frac{1}{2})q/x_0}\right)}} = \underline{\underline{\sinh\left(\frac{q}{2x_0}\right)e^{-kq/x_0}}}$$

If $k = 0$, $p_0 = 2\int_0^{\frac{1}{2}q} p(x)\,dx = \underline{\underline{\left(1 - e^{-\frac{1}{2}q/x_0}\right)}}$

If $k < 0$, $p_k = p_{-k}$

$$\therefore\ p_k = \underline{\underline{\sinh\left(\frac{q}{2x_0}\right)e^{-|k|q/x_0}}} \qquad \text{if } k \neq 0$$

$$\&\ \ p_0 = \underline{\underline{1 - e^{-q/2x_0}}}$$

4. (cont) Variance, $\sigma^2 = \sum_{k=-\infty}^{\infty} (kq)^2 p_k$ (mean = 0 by symmetry)

$$= 2 \sum_{k=1}^{\infty} (kq)^2 \sinh\left(\frac{q}{2x_0}\right) e^{-kq/x_0}$$

Now if $S = \sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$

$$\frac{dS}{dr} = \sum_{k=1}^{\infty} k r^{k-1} = \frac{1}{(1-r)^2}$$

$$\therefore \sum_{k=1}^{\infty} k r^k = \frac{r}{(1-r)^2} = \frac{1}{(1-r)^2} - \frac{1}{1-r}$$

Diff again dr: $\sum_{k=1}^{\infty} k^2 r^{k-1} = \frac{2}{(1-r)^3} - \frac{1}{(1-r)^2} = \frac{1+r}{(1-r)^3}$

$$\therefore \sum_{k=1}^{\infty} k^2 r^k = \frac{r(1+r)}{(1-r)^3}$$

$\therefore \sigma^2 = 2q^2 \cdot \sinh\left(\frac{q}{2x_0}\right) \cdot \frac{r(1+r)}{(1-r)^3}$ where $r = e^{-q/x_0}$

Entropy

$$H = -\sum_{k=-\infty}^{\infty} p_k \log_2 p_k$$

$$= -p_0 \log_2 p_0 - 2\sum_{k=1}^{\infty} p_k \log_2 p_k \quad \text{by symmetry.}$$

Consider the summation & let $p_k = \alpha e^{\beta k}$ $\left(\begin{array}{l}\alpha = \sinh\left(\frac{q}{2x_0}\right) \\ \beta = -q/x_0\end{array}\right)$

$$\therefore \sum_{k=1}^{\infty} \alpha e^{\beta k} \log_2 \alpha e^{\beta k} = \frac{\alpha}{\ln 2} \sum_{k=1}^{\infty} e^{\beta k} (\ln\alpha + \beta k)$$

$$= \frac{\alpha}{\ln 2}\left[\ln\alpha . \sum_{k=1}^{\infty} e^{\beta k} + \beta \sum_{k=1}^{\infty} k e^{\beta k}\right]$$

$$= \frac{\alpha \ln\alpha}{\ln 2} \cdot \frac{e^{\beta}}{1-e^{\beta}} + \frac{\alpha\beta}{\ln 2} \cdot \frac{e^{\beta}}{(1-e^{\beta})^2} = \frac{\alpha e^{\beta}}{\ln 2 (1-e^{\beta})^2}\left[\ln\alpha(1-e^{\beta}) + \beta\right]$$

4. (cont) Consider the summation & let $p_k = \alpha r^k$ where $\alpha = \sinh\left(\frac{q}{2x_0}\right)$ & $r = e^{-q/x_0}$

$$\therefore \sum_{k=1}^{\infty} \alpha r^k . \log_2(\alpha r^k) = \alpha \sum_{k=1}^{\infty} r^k (\log_2 \alpha + k \log_2 r)$$

$$= \alpha \log_2 \alpha \sum_{k=1}^{\infty} r^k + \alpha \log_2 r \sum_{k=1}^{\infty} k r^k$$

$$= \alpha \log_2 \alpha . \frac{r}{1-r} + \alpha \log_2 r . \frac{r}{(1-r)^2} *$$

$$= \frac{\alpha r}{(1-r)^2}\left[(1-r)\log_2 \alpha + \log_2 r\right]$$

$$p_0 \log_2 p_0 = (1-\sqrt{r}) . \log_2(1-\sqrt{r})$$

$$\therefore \text{Entropy, } H = -(1-\sqrt{r}) \log_2 (1-\sqrt{r}) - \frac{2\alpha r}{(1-r)^2}\left[(1-r)\log_2 \alpha + \log_2 r\right]$$

The variance of the pdf gives the energy per pel in the sub-image and is a direct measure of the energy compression properties of the transform being used.

The entropy estimates the bit rate required to code the quantised values and is more directly useful than energy for assessing transform effectiveness in a coding application.

5. For $p_0 = 0.8$: $$1 - e^{-q/2x_0} = 0.8$$

$$\therefore \frac{q}{2x_0} = -\ln 0.2 = 1.6094 \qquad \therefore q = \underline{\underline{3.219\,x_0}}$$

$$\alpha = \sinh\left(\frac{q}{2x_0}\right) = \tfrac{1}{2}\left(5 - 0.2\right) = 2.4$$

$$\tau = e^{-q/x_0} = 0.2^2 = 0.04$$

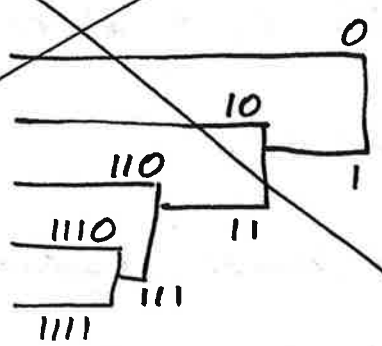
$$p_k = \alpha\,\tau^k \qquad \therefore \quad p_1 = 0.096$$
$$p_2 = 0.00384$$
$$p_3 = 0.0001536$$

Prob of state 4 or greater $= \sum_{k=4}^{\infty} p_k = p_4 \sum_{k=0}^{\infty} \tau^k = \frac{p_4}{1-\tau}$

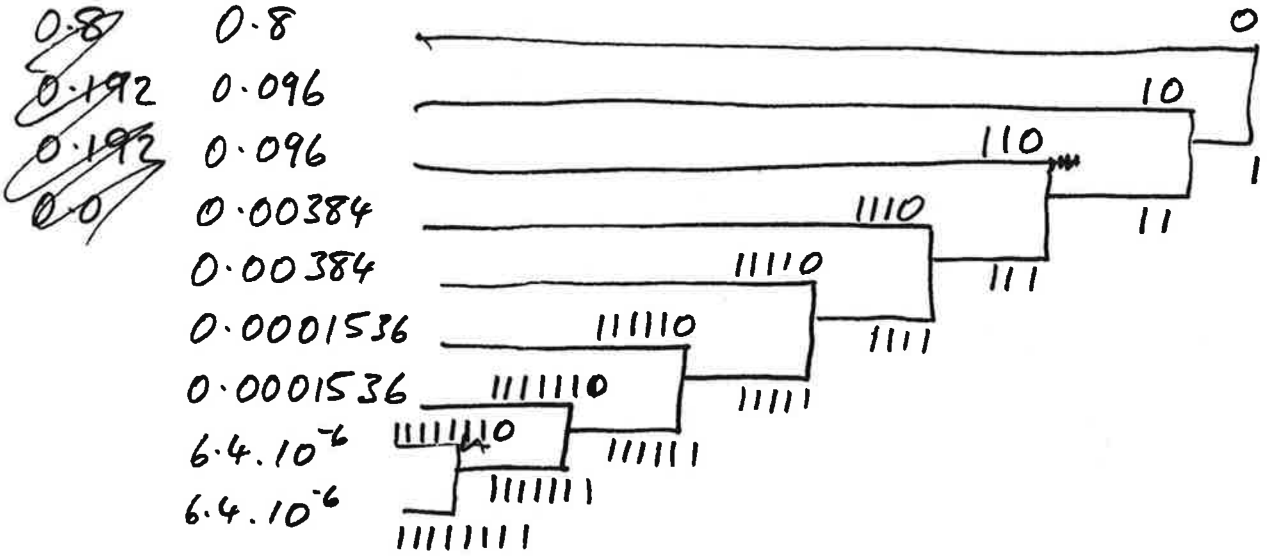
$$= 6.4 \cdot 10^{-6}$$

~~To simplify Huffman code, encode $|k|$ & then add an extra sign bit if $k \neq 0$. $\therefore$ We double the probabilities for p_k, $k \neq 0$.~~

k	p_k
0	0.8
1	0.192
2	0.00768
3	0.0003072
≥ 4	$1.28 \cdot 10^{-5}$



5. (cont)



Entropy for states $|k| \geq 4$

$$= -2 \sum_{k=4}^{\infty} \alpha r^k \log_2(\alpha r^k)$$

This can be ignored for $|k| > 4$ since r is so small, so use actual code lengths.

$\therefore$ Mean bit rate $= 0.8 . 1 + 0.096(2+3) + 0.00384(4+5)$
$+ 0.0001536(6+7) + 6.4.10^{-6}(8+9) + \ldots$

$= 1.3167$ bit/pel.

Using result from previous question:

$$\text{Entropy, } H = -\left\{0.8 \log_2 0.8 - \frac{2 . 2.4 . 0.04}{0.96^2}\left[0.96 \log_2 2.4 + \log_2 0.04\right]\right\}$$

$= 0.2575 - 0.2083[1.2125 - 4.6439]$
$= 0.9723$ bit/pel.

$$\text{Efficiency} = \frac{0.9723}{1.3167} = \underline{73.8\%}$$

5. (cont) The main source of inefficiency is the coding of p_0, which in theory only needs $-\log_2 0.8 = 0.322$ bits but in practice has to use 1 bit. This is compensated to some extent by shorter words than $-\log_2 p_k$ for the other states.

To improve the efficiency, run-length coding (RLC) of the p_0 state could be used, which would:

a) reduce no. of events to be coded.

b) separate the p_0 ~~state~~ event into many less probable events, corresponding to each length of run of zeros.

6.

Non-zero coef	(Run, Size)	Huffman code	Additional Bits
11	4 (DC table)	101	1011
−9	0,4	1011	0110
−7	0,3	100	000
2	0,2	01	10
3	1,2	11011	11
1	0,1	00	1
−1	7,1	11111010	0
1	4,1	111011	1
EOB	0,0	1010	–

∴ Final code is:

101 1011 1011 0110 100 000 01 10 11011 11
00 1 11111010 0 111011 1 1010

No. of bits = 55 ∴ Bit rate = $\frac{55}{64}$ = <u>0.86 bit/pel</u>

7. Transfer function to bandpass output at level k is:

$$H_{k,1}(z) = \prod_{i=1}^{k-1} H_0\left(z^{2^{i-1}}\right) \cdot H_1\left(z^{2^{k-1}}\right)$$

$$H_{k-1,1}(z^2) = \prod_{i=1}^{k-2} H_0\left((z^2)^{2^{i-1}}\right) \cdot H_1\left((z^2)^{2^{k-2}}\right)$$

Now $\left(z^2\right)^{2^{i-1}} = z^{2.2^{i-1}} = z^{2^i}$

$$\therefore H_{k-1,1}(z^2) = \prod_{i=1}^{k-2} H_0\left(z^{2^i}\right) \cdot H_1\left(z^{2^{k-1}}\right)$$

$$= \prod_{i=2}^{k-1} H_0\left(z^{2^{i-1}}\right) \cdot H_1\left(z^{2^{k-1}}\right)$$

$$\therefore H_{k,1}(z) = H_0(z) \cdot H_{k-1,1}(z^2)$$

Now $H_{1,1}(z) = H_1(z) = \frac{1}{8} z^{-1}\left(-z^2 - 2z + 6 - 2z^{-1} - z^{-2}\right)$

$\therefore H_{2,1}(z) = H_0(z) . H_{1,1}(z^2) = \frac{1}{2}(z + 2 + z^{-1}) . \frac{1}{8} z^{-2}\left(-z^4 - 2z^2 + 6 - 2z^{-2} - z^{-4}\right)$

$= \frac{1}{16} z^{-2}\left(-z^5 - 2z^4 - 3z^3 - 4z^2 + 4z + 12 + 4z^{-1} \ldots - z^{-5}\right)$ (symmetric)

$H_{3,1}(z) = H_0(z) . H_{2,1}(z^2) = \frac{1}{2}(z + 2 + z^{-1}) . \frac{1}{16} z^{-4}\left(-z^{10} - 2z^8 - 3z^6 \ldots\right)$

$= \frac{1}{32} z^{-4}\left(-z^{11} - 2z^{10} - 3z^9 - 4z^8 - 5z^7 - 6z^6 - 7z^5 - 8z^4 + 0.z^3\right.$
$\left. + 8z^2 + 16z + 24 + 16z^{-1} + \ldots\ldots - z^{-11}\right)$

7. (cont.) For the inverse filter pair:

$$H_{2,1}(z) = H_0(z) \,.\, H_1(z^2) = \tfrac{1}{8}\left(-z^2+2z+6+2z^{-1}-z^{-2}\right) . \tfrac{1}{2} z^2\left(-z^2+2-z\right.$$

$$= \tfrac{1}{16} z^2\left(z^4 - 2z^3 - 8z^2 + 2z + 14 + 2z^{-1} - 8z^{-2} - 2z^{-3} + z^{-4}\right)$$

$$H_{3,1}(z) = H_0(z) \,.\, H_{2,1}(z^2)$$

$$= \tfrac{1}{8}\left(-z^2+2z+6+2z^{-1}-z^{-2}\right) . \tfrac{1}{16} z^4\left(z^8 - 2z^6 - 8z^4 + 2z^2 + 14 + \ldots\right)$$

$$= \tfrac{1}{32} z^4\left(-z^{10} + 2z^9 + 8z^8 - 2z^7 - 5z^6 - 20z^5 - 48z^4 - 12z^3 \right.$$
$$\left. + 6z^2 + 32z + 80 + 32z^{-1} + \ldots \; - z^{-10}\right)$$

The first pair gives much smoother impulse responses (comprising linear ramps), so these should be used for image reconstruction, where smoothness is much more critical.

$\therefore$ Best arrangement is to use the 2^{nd} pair for analysis & the first pair for reconstruction.

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8. $$(1+Z)(1+aZ+bZ^2) = 1 + (1+a)Z + (a+b)Z^2 + bZ^3$$
$$(1+Z)(1+cZ) = 1 + (1+c)Z + cZ^2$$

$$\therefore P_t(Z) = 1 + (2+a+c)Z + \left(c + (1+a)(1+c) + a + b\right)Z^2$$
$$+ (\ldots)Z^3 + \left(c(a+b) + b(1+c)\right)Z^4 + bcZ^5$$

For PR, terms in Z^2 & Z^4 must be zero.

$$\therefore\; a+b+c+(1+a)(1+c) = 0 \qquad — (1)$$
$$\& \;\; c(a+b) + b(1+c) = 0 \qquad — (2)$$

8. (cont) From ①: $b = -(1 + 2a + 2c + ac)$

Subst in ②: $-c(1 + a + 2c + ac) - (1+c)(1 + 2a + 2c + ac) = 0$

$$\therefore \quad a(c + c^2 + 2 + 2c + c + c^2) + c + 2c^2 + 1 + 3c + 2c^2 = 0$$

$$\therefore \quad 2a(1 + 2c + c^2) + 1 + 4c + 4c^2 = 0$$

$$\therefore \quad \underline{\underline{a = -\frac{(1+2c)^2}{2(1+c)^2}}}$$

$$b = -\left(1 + 2c + a(2+c)\right) = \frac{-(1+2c)}{2(1+c)^2}\left[2(1+c)^2 - (1+2c)(2\ \right]$$

$$= \frac{-(1+2c)}{2(1+c)^2}\left[2 + 4c + 2c^2 - 2 - 5c - 2c^2\right] = \underline{\underline{\frac{c(1+2c)}{2(1+c)^2}}}$$

At $Z = -1$, $Z + 1 = 0$ so both factor pairs are zero

At $Z = 0$, both factor pairs $= 1$

At $Z = 1$, L.H. pair $= 2(1 + a + b)$
R.H. pair $= 2(1 + c)$

Hence if $c = a + b$, the requirement is met.

$$\therefore c = \frac{1}{2(1+c)^2}\left(c(1+2c) - (1+2c)^2\right)$$

$$\therefore \quad 2c(1+c)^2 = -1 - 3c - 2c^2 = -(1+c)(1+2c)$$

$c = -1$ is not sensible since this would make a & $b = \infty$

$$\therefore \quad 2c(1+c) + 1 + 2c = 0 \qquad \therefore \quad 2c^2 + 4c + 1 = 0$$

$$\therefore \quad c = \frac{-4 \pm \sqrt{16-8}}{4} = -1 \pm \sqrt{\tfrac{1}{2}} = -0.2929 \text{ or } -1.7071$$

8. (cont) In order to make the L.H. factor pair behave like a lowpass filter when $Z = \cos(\omega T_s)$, c should be as small in magnitude as possible.

$\therefore$ Choose $c = -1 + \frac{1}{\sqrt{2}} = -0.2929$.

~~or~~ $-\frac{2}{7} = -0.2857$ which is pretty close

and has the advantage that the coefs of the two filters are then rational numbers if $c = -\frac{2}{7}$. This makes it easier to implement the filters exactly with arithmetic circuits of limited word length (as in DSP chips).

The range $Z = -1 \rightarrow +1$ corresponds to all valid values of Z when $Z = \cos(\omega T_s)$.

Hence if the two factor pairs are similar over $Z = -1 \rightarrow +1$, their frequency responses will be similar and the reconstruction filters will be similar to the analysis filters, which is the ideal situation for filter banks.

9. When $Z=-1$: $\quad pz^3+\left(\frac{1}{2}-p\right)\left(z+z^{-1}\right)+pz^{-3}+1=0$

This is symmetric about z^0, so ~~the~~ factors must also be expressable as symmetric functions about z^0.

Factorise out the symmetric double zero at $z=-1$: $(z+1)(1+z^{-1})$

$$\frac{pz^3+\left(\frac{1}{2}-p\right)\left(z+z^{-1}\right)+pz^{-3}+1}{z+2+z^{-1}} = pz^2-2pz+\left(\tfrac{1}{2}+2p\right)-2pz^{-1}+pz^{-2}$$

For there to be >2 zeros at $z=-1$, RHS above must be zero when $z=-1$.

$$\therefore\quad p+2p+\left(\tfrac{1}{2}+2p\right)+2p+p=0$$
$$\therefore\ 8p=-\tfrac{1}{2} \qquad \text{so } \underline{\underline{p=-\tfrac{1}{16}}}$$

There are then 4 zeros at $z=-1$, since the factors must be symmetric. The remaining factor is $(-z+4-z^{-1})$ which has no more zeros at $z=-1$.

Letting $z=e^{j\omega T_s}$:

$$Z=(1-2p)\cos(\omega T_s)+2p\cos(3\omega T_s)$$

$$\frac{dZ}{d\omega}=-T_s(1-2p)\sin(\omega T_s)-6T_sp\sin(3\omega T_s)$$
$$=0 \quad \text{if } \omega T_s=\pi$$

$$\frac{d^2Z}{d\omega^2}=-T_s^2(1-2p)\cos(\omega T_s)-18T_s^2p\cos(3\omega T_s)$$
$$=T_s^2(1-2p+18p)=0 \quad \text{if } \omega T_s=\pi \text{ \& } p=\tfrac{-1}{16}$$

9. (cont) $\frac{d^3 Z}{d\omega^2} = T_s^3(1-2p)\sin(\omega T_s) + 54\, T_s^3 p \sin(3\omega T_s)$
$= 0 \quad \text{if } \omega T_s = \pi$

$$\frac{d^4 Z}{d\omega^4} = T_s^4(1-2p)\cos(\omega T_s) + 162\, T_s^3 p \cos(3\omega T_s)$$
$$\neq 0 \quad \text{when } \omega T_s = \pi \ \& \ p = -\tfrac{1}{16}$$

∴ The first 3 derivatives are zero & all odd derivatives are zero beyond that.

∴ $(Z+1)$ is a very flat function of ω near $\omega T_s = \pi$, so each $(Z+1)$ factor will remain very small over quite a wide range of frequencies near $\omega T_s = \pi$.

If $p = \frac{-1}{16}$, $Z = \frac{1}{16}\left(-z^3 + 9z + 9z^{-1} - z^{-3}\right)$

$$\therefore \quad Z^2 = \frac{1}{256}\left(z^6 - 18z^4 + 63z^2 + 164 + 63z^{-2} - 18z^{-4} + z^{-6}\right)$$

$$\& \quad Z^3 = \frac{1}{4096}\left(-z^9 + 27z^7 - 216z^5 + 240z^3 + 1998z^1 + 1998z^{-1} + 240z^{-3} - 216z^{-5} + 27z^{-7} - z^{-9}\right)$$

$$G_0(z) = \frac{1}{7}(7 + 5Z - 2Z^2) = \cancel{\frac{1}{7.2048}\left(2048 \ldots\right.}$$

$$= \frac{1}{7.128}\left(7.128 + 5.128\,Z - 256\,Z^2\right)$$

$$= \frac{1}{\cancel{7.128}\ 896}\left(-z^6 + 18z^4 - 40z^3 - 63z^2 + 360z + 732 + 360z^{-1} - \ldots - z^{-6}\right)$$

$$H_0(z) = \frac{1}{50}\left(50 + 41Z - 15Z^2 - 6Z^3\right)$$
$$= \frac{1}{50.2048}\left(50.2048 + 41.128.16Z - 15.8.256Z^2 - 3.4096Z^3\right)$$

$$= \frac{1}{102400}\left(3z^9 - 81z^7 - 120z^6 + 648z^5 + 2160z^4 - 5968z^3 - 7560z^2 + 41{,}238z + 82{,}720 + 41{,}238z^{-1} - \ldots + 3z^{-9}\right)$$

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