How to solve 1B PDEs

Pete Bunch

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Most partial differential equations do not have analytic solutions (i.e. there isn't an equation for it). For those that do, there are dozens of different methods. However, for 1B engineering, we only look at the (quite small) class of PDEs with separable solutions, i.e. if $f(x, y) = X(x)Y(y)$.

1 The Method

1.1 Step 1: Apply the separability condition

$$f(x, y) = X(x)Y(y) \tag{1}$$

1.2 Step 2: Rearrange it

Shift terms around in order to get your PDE in the following form:

$$\text{things with } x = \text{things with } y \tag{2}$$

If we can't get it into this form, then the PDE cannot be solved by the separation of variables method.

Now, if we were to change only x or only y, we'd be affecting only one side of the equality, so the only way this can be true is if both sides are equal to a constant,

$$\text{things with } x = \text{things with } y = k. \tag{3}$$

We can solve this as two separate equations.

$$\begin{aligned} &\text{things with } x = k \\ &\text{things with } y = k \end{aligned} \tag{4}$$

These are both ordinary differential equations.

1.3 Step 3: Solve the two differential equations

We have two ODEs to solve. If either ODE doesn't have a solution, then neither does the PDE. In part 1A, you learned to solve constant coefficient linear ODEs, so lets hope they both fall into that class. If they don't, the question will usually give you a trial solution (e.g. Q1 on the examples paper). Furthermore, they'll usually be first or second order only.

Remember that when we solve an constant coefficient linear ODE, there are many possible functional forms for the solution: exponential, sinusoidal, linear or any combination of the these. Which of these it is will depend on the constant, k, which we introduced above. By comparing the functional form of the boundary conditions, we can often identify the sign, and sometimes the exact value, of k. This will make it easier to solve the second equation.

1.4 Step 4: Put it together

Multiply the two ODE solutions together to get the solution to the PDE. Use the boundary conditions to find the remaining arbitrary constants.

2 Worked example

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 f}{\partial x^2} + 2\omega \frac{\partial f}{\partial x} \tag{5}$$

for $x \geq 0$ and positive real constant, ω, with boundary conditions

$$f(0,t) = \sin(\omega t) \tag{6}$$

$$\frac{\partial f}{\partial x}(0,t) = 0 \tag{7}$$

2.1 Step 1

$$f(x,t) = X(x)T(t) \tag{8}$$

so

$$XT'' = X''T + 2\omega X'T. \tag{9}$$

2.2 Step 2

$$\frac{T''}{T} = \frac{X'' + 2\omega X'}{X} = k \tag{10}$$

2.3 Step 3

First the T equation:

$$\begin{aligned} \frac{T''}{T} &= k \\ T'' - kT &= 0 \end{aligned} \tag{11}$$

Auxiliary equation:

$$\lambda^2 - k = 0 \tag{12}$$

so

$$\lambda = \pm\sqrt{k}. \tag{13}$$

This gives us three possible solutions:

$$\begin{aligned} T(t) &= A\exp\{\sqrt{k}t\} + B\exp\{-\sqrt{k}t\}, && k > 0 \\ T(t) &= A\cos(\sqrt{|k|}t) + B\sin(\sqrt{|k|}t), && k < 0 \\ T(t) &= A + Bt, && k = 0 \end{aligned} \tag{14}$$

The only way any of these can match the $\sin(\omega t)$ in the boundary conditions is if $k < 0$ and $|k| = \omega^2$. Thus $k = -\omega^2$. Furthermore, we need $A = 0$ to get rid of the cos term. So that gets us down to

$$T(t) = B\sin(\omega t), \qquad k = -\omega^2 \tag{15}$$

Now for the X equation:

$$\begin{aligned} \frac{X'' + 2\omega X'}{X} &= k \\ X'' + 2\omega X' - kX &= 0 \\ X'' + 2\omega X' + \omega^2 X &= 0 \end{aligned} \tag{16}$$

Auxiliary equation:

$$\begin{aligned} \lambda^2 + 2\omega\lambda + \omega^2 &= 0 \\ (\lambda + \omega)^2 &= 0 \end{aligned} \tag{17}$$

so

$$\lambda = -\omega. \tag{18}$$

and this is a repeated root, so the solution is

$$X(x) = (C + Dx)\exp\{-\omega x\} \tag{19}$$

2.4 Step 4

Complete solution:

$$\begin{aligned} f(x,t) = X(x)T(t) &= B\sin(\omega t)(C + Dx)\exp\{\omega x\} \\ &= (E + Fx)\exp\{-\omega x\}\sin(\omega t) \end{aligned} \tag{20}$$

where E and F are new arbitrary constants composed of the product of the old arbitrary constants.

Finally, using the first boundary conditions,

$$\begin{aligned} \frac{\partial f}{\partial x}(0,t) &= [(E + Fx)(-\omega)\exp\{-\omega x\} + F\exp\{-\omega x\}]\sin(\omega t)|_{x=0} \\ &= -E\omega + F \\ &= 0. \end{aligned} \tag{21}$$

so $F = E\omega$.

$$f(x,t) = X(x)T(t) = E(1 + \omega x)\exp\{-\omega x\}\sin(\omega t) \tag{22}$$

Second boundary condition

$$\begin{aligned} f(0,t) &= E\sin(\omega t) \\ &= \sin(\omega t) \end{aligned} \tag{23}$$

so $E = 1$.

Final solution:

$$f(x,t) = (1 + \omega x)\exp\{-\omega x\}\sin(\omega t) \tag{24}$$

Bosh.